MA 200 - Lecture 23

1 Recap

1. Orientations for 1-manifolds through unit tangent vector fields.
2. Orientations for hypersurfaces through unit normal vector fields.
3. Restrictions of orientations from M to ∂M produce orientations for ∂M. A standard orientation for ∂M is the restriction if $dim(M)$ is even, and the opposite of the restriction when $dim(M)$ is odd.

2 Differential forms, wedge products, and form-fields in $\mathbb{R}^n$

We want to generalise the notion of a cross product (because that will also help us generalise the notion of curl). Naively, if $a, b \in \mathbb{R}^4$, then $a \times b$ ought to have components $a_i b_j - a_j b_i$ where $i \neq j$. Even if we demand $i < j$ (because anyway, $a_i b_j - a_j b_i = -(a_j b_i - a_i b_j)$, i.e., $a \times b = -b \times a$), the number of components is 6. So $a \times b$ cannot be a vector in $\mathbb{R}^4$! (On the other hand, $a \times b \times c$ in this naive prescription will have only 4 components!) So we need to extend our definitions from vectors to other beasts. Whatever this naive $a \times b$ is, each component is certainly multilinear in a, b and antisymmetric. That motivates the following definitions.
Let V be a f.d real vector space. We will almost always consider $V = \mathbb{R}^k$ or $T_p M$ for some manifold-with-boundary. We know what a multilinear map $T : V \times V \times \ldots \to \mathbb{R}$ is (what is it?) Multilinear maps from V^m to $\mathbb{R}$ are also called m-tensors. An example is $T(\vec{v}, \vec{w}) = \det(\vec{v}\ \vec{w})$ where $\vec{v}, \vec{w} \in \mathbb{R}^2$. Another is $p(\vec{v}, \vec{w}) = v_1 w_1 + v_2 w_2 = \langle v, w \rangle$. The set of m-tensors forms a vector space. We can find a nice basis for this vector space: Given an m-multiindex $I = (i_1, \ldots, i_m)$ and a basis $e_1, \ldots, e_n$, the tensors $\phi_I(v_1, \ldots, v_m) = (v_1)_{i_1} (v_2)_{i_2} \ldots$ form a basis (so the dimension is n^m where $n = dim(V)$). By the way, for a single index, the 1-tensors ϕ_i are called the dual basis of e_j. 1-tensors are also called linear functionals, or as we shall call them later, 1-forms. (By the 0-forms are simply real numbers.)
A neat way of phrasing this statement is through the definition of the tensor product of two tensors: Suppose S, T are k and l tensors respectively. The map $S \otimes T$ defined as $S \otimes T(v_1, \ldots, v_k, w_1, \ldots, w_l) = S(v)T(w)$ is a $(k+l)$-tensor called the tensor product of S and T. Note that $\phi_I = \phi_{i_1} \otimes \phi_{i_2} \ldots$. This tensor product obeys some standard properties, namely, associativity and linearity.

Given a linear map $T : V \to W$, there is a 'dual' linear map denoted as T^* from k-tensors on W to those on V: $T^*L(v_1, \ldots, v_k) = L(Tv_1, \ldots, Tv_k)$. This dual map obeys some (easy to verify) properties: T^* is linear on the space of k-tensors on W, $T^*(f \otimes g) = T^*f \otimes T^*g$ and $(S \circ T)^* = T^* \circ S^*$.
What we want to study are tensors that are *antisymmetric* or *alternating* (akin to our example of determinants, or cross products):
Def: A k-tensor is said to be *symmetric* if $f(v_1, \ldots, v_k) = f(v_1, \ldots, v_{i+1}, v_i, \ldots)$, i.e., interchanging adjacent arguments doesn't change the value. By a finite number of *adjacent* interchanges, one can see that interchanging *any* two arguments keeps the value invariant. Since every permutation $\sigma \in S_k$ is a product of transpositions (hopefully you did this in UM 205), the value is invariant under any permutation of the arguments. A k-tensor is said to be *antisymmetric* or *alternating* $f(v_1, \ldots, v_k) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots)$. Such tensors are also called k-forms and they form a vector space that is denoted as $\Lambda^k V$. (For $k = 1$, every tensor is trivially alternating.) They are also called k-forms. Clearly, if $v_i = v_j$ for some $i \neq j$, the value of f is zero on such a tuple (if $|j - i| = 1$, this is trivial. Now induct on $|j - i|$). Moreover, if a tensor is such that it is zero whenever adjacent arguments coincide, then it is alternating: $0 = f(v_1, \ldots, v_i + v_{i+1}, v_i + v_{i+1}, v_{i+2}, \ldots)$. Now use multi-linearity and the vanishing property again to be done.
Inner products are examples of symmetric 2-tensors and the determinant an example of $\Lambda^n(\mathbb{R}^n)$. In fact, we can produce alternating 3-tensors in $\mathbb{R}^4$ using the determinant as $(u, v, w) \to \det(e_1\ u\ v\ w)$ and so on. So what is the dimension of $\Lambda^k V$? Suppose $e_1, \ldots, e_n$ is a basis of V. Define the following alternating k-tensors for a given k-multiindex $I = (i_1, \ldots, i_k)$:

$$\epsilon_I(v_1, \ldots, v_k) = \det(A_I)\ where\ (A_I)_{jl} = (v_l)_{i_j}. \tag{1}$$

Here are examples:

1. When $k = 1$, ϵ_i form the dual basis for $V^* = \Lambda^1 V$.
2. Suppose $k = 2$ and $I = (2, 3)$, $v_1 = (a, b, c)$, $v_2 = (\alpha, \beta, \gamma)$ then $\epsilon_I(v_1, v_2) = \det \begin{pmatrix} b & \beta \\ c & \gamma \end{pmatrix} = b\gamma - c\beta$.
3. Suppose $i_1 = i_3$. Then $\epsilon_I = 0$ (why?) More generally, when two indices coincide, $\epsilon_I = 0$.
4. Recall that the sign of a permutation $\sigma \in S_k$ is simply the determinant of the corresponding permutation matrix. Now we define $\sigma(I) = (i_{\sigma(1)}, \ldots)$. (For instance, if $k = 3$, and $\sigma = (12)$, then $\sigma(2, 3, 1) = (3, 2, 1)$. Then $\epsilon_{\sigma(I)} = sgn(\sigma)\epsilon_I$ by the properties of determinants.

Now we consider the multiindices I such that $i_1 < i_2 < \ldots < i_k$. There are $\binom{n}{k}$ of these indices. We claim that such ϵ_I form a basis for $\Lambda^k(V)$. Indeed,

1. These ϵ_I are linearly independent: Suppose $\sum_{i_1 < \ldots < i_k} c_I \epsilon_I = 0$. Then if $j_1 < j_2 < \ldots < j_k$, $0 = \sum c_I \epsilon_I(e_{j_1}, \ldots, e_{j_k}) = c_J$ (why?)

2. They span $\Lambda^k(V)$: Let $\omega \in \Lambda^k(V)$. Then consider $\tilde{\omega} = \sum \omega(e_{i_1}, \ldots, e_{i_k})\epsilon_I$. Note that to prove that $\tilde{\omega} = \omega$, it is enough to show that they are equal on $(e_{j_1}, \ldots, e_{j_k})$ for all increasing multiindices J (why? Because of multilinearity, and the facts that if two indices coincide, we get zero and if we change the ordering, we pick up the sign of the permutation). Now $\tilde{\omega}(e_J) = \sum \omega(e_I)\epsilon_I(e_J) = \omega(e_J)$ (where we abuse notation by denoting a tuple $(w_{i_1}, \ldots)$ by w_I).

Note that when $k = n$, Λ^n has dimension 1 and is generated by $\epsilon_{12\ldots n}$. These forms are also called 'top forms'. When $k = 0$, again the dimension is 1. For $n = 3$, Λ^1 and Λ^2 have exactly the same dimension equal to 3 (which is also the dimension of V!) So we can identify a vector $v \in \mathbb{R}^3$ with a 1-form $\omega = v_1\epsilon_1 + v_2\epsilon_2 + v_3\epsilon_3$ and with a 2-form $v_1\epsilon_{23} + v_2\epsilon_{31} + v_3\epsilon_{12}$ (why this weird identification? Because we want to think of ϵ_{12} as $\hat{i} \times \hat{j} = \hat{k}$).
We are now in a position to define the generalisation of the cross product. Instead of defining it for vectors, we define the wedge product $\wedge$ of forms. We want to try the following naive definition: Let $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. Then $\omega \wedge \eta \in \Lambda^{k+l}(V)$ is defined as $\omega \wedge \eta = \sum_{i_1 < \ldots < i_k, j_1 < \ldots < j_l} \omega_I \eta_J \epsilon_{IJ}$, i.e., we define $\epsilon_I \wedge \epsilon_J$ as ϵ_{IJ} and extend this definition linearly.