MA 200 - Lecture 22

1 Recap

1. Proved that it is okay to cover a manifold with coordinate parametrisations upto measure zero for the purposes of finding areas/volumes/integrating functions.

2. Sketch of a proof of Green's theorem.

3. Defined orientability for dimension ≥ 2 and saw how it could potentially be problematic for dimension $= 1$.

4. Stated a theorem about the existence of unit normal vector fields for orientable $(n-1)$-manifolds in $\mathbb{R}^n$.

2 Orientability of manifolds

1. Here is a lemma that produces several examples: Let M be an $n-1$-dimensional (where $n-1 \geq 2$) manifold in $\mathbb{R}^n$. A C^r-varying unit normal vector field on M is a function $n : M \to \mathbb{R}^n$ such that for any coordinate patch α, $n \circ \alpha$ is C^r (by the chain rule, this can be accomplished by making sure such is the case for *some* collection of patches that cover M), $n(p) \perp T_pM \ \forall \ p$. Now M is orientable iff it has a C^r-varying unit normal vector field:

   (a) M is orientable: Consider a cover by orientation-compatible coordinate parametrisations α_i. Then note that at each point, there are only two choices of unit normals (why?) we make the choice by requiring that $\det(\vec{n}(p)\ \frac{\partial \alpha_i}{\partial u_1}(\alpha^{-1}(p))\ \frac{\partial \alpha_i}{\partial u_2}\alpha^{-1}(p)\ \ldots) > 0$. In the HW you will show that such an $\vec{n}$ is C^r by producing an explicit formula for it.

   (b) There is a C^r-varying unit normal vector field $\vec{n}$: Consider all parametrisations α such that $\det(\vec{n}(p)\ \frac{\partial \alpha_i}{\partial u_1}(\alpha^{-1}(p))\ \frac{\partial \alpha_i}{\partial u_2}(\alpha^{-1}(p))\ \ldots) > 0$. They form an orientation. Indeed, they exist: If we take any parametrisation near p such that this condition fails at p, simply reverse it (and the condition will be met in a neighbourhood of p). They are compatible: Use the chain rule (HW).

   In other words, if we take a surface in $\mathbb{R}^3$ defined by $f = 0$ where $\nabla f \neq 0$ on $f = 0$, then it is orientable. Concretely, we can take say, the sphere S^2, and explicitly produce orientated coordinate parametrisations.

2. Suppose $M \subset \mathbb{R}^3$ is an 3-dimensional manifold-with-boundary given by $f \leq 0$ where $f : \mathbb{R}^3 \to \mathbb{R}$ is C^r and $\nabla f \neq 0$ on $f = 0$. Then cover $f < 0$ by the usual coordinate chart. Near ∂M, consider $\vec{n} = \frac{\nabla f}{\|\nabla f\|}$. Note that the boundary charts $(y_1, y_2, y_3) = (-x_2, x_3, -f)$ (if $f_1 > 0$), $(x_1, x_3, -f)$ (if $f_2 > 0$), $(-x_1, x_2, -f)$ (if $f_3 > 0$), etc are orientation-compatible with each other and the usual interior chart. Moreover, the restrictions on the boundary, i.e., to $f = 0$ form an orientation for the boundary (why?).

3. More generally, if M is an oriented manifold-with-nonempty-boundary, then ∂M is orientable and inherits a standard god-given orientation (that coincides in the case of domains in $\mathbb{R}^2$ with the orientation needed for Green's theorem): Consider boundary charts $(u_1, \ldots, u_k) \in \mathbb{H}^k \to \alpha(u) \in \mathbb{R}^n$ that are orientation-compatible. Restrict them to the boundary, i.e., $(v_1, \ldots, v_{k-1}) \to \alpha(v_1, \ldots, v_{k-1}, 0)$. Suppose β is another parametrisation such that on the overlap, the transition map $g = \beta^{-1} \circ \alpha$ is orientation preserving on an open subset of $\mathbb{H}^k$. On $\partial \mathbb{H}^k$, the last row of Dg is $Dg_k = (0, 0, 0 \ldots, 0, \frac{\partial g_k}{\partial x_k})$. Since $\det(Dg) > 0$, $\frac{\partial g_k}{\partial x_k} \neq 0$. It is in fact, > 0 because if we move in the direction of x_k, i.e., into the open set, then we will do so in the image too. Thus, $\det(\frac{\partial(g_1, \ldots, g_{k-1})}{\partial(x_1, \ldots, x_{k-1})}) > 0$ and this is precisely the Jacobian of the restriction of the coordinate parametrisations. We say that ∂M has the *induced* orientation from M if $dim(M)$ is even and we equip ∂M with the orientation given by the restrictions of these charts. If $dim(M)$ is odd, we choose the opposite orientation. This god-given orientation *coincides* (HW) with the following: Parametrisations $\gamma(u_1, \ldots, u_{k-1})$ of the boundary such that at every point $[\vec{n}(p) \, \frac{\partial \gamma}{\partial u_1}(p) \, \frac{\partial \gamma}{\partial u_2}(p) \ldots]$ is orientation compatible with the basis $[\frac{\partial \alpha}{\partial u_i}]$ where α is an oriented parametrisation, and $n(p)$ an the outward-pointing vector, $\vec{n}(p) = -\frac{\partial \alpha}{\partial x_n}(\alpha^{-1}(p))$.
So for instance, in Green's theorem, moving "anticlockwise" along the "outer boundary", i.e., moving along the boundary (with your head held high because you are doing maths) such that the region lies to your left, means that $outward\ normal \ \times \ tangent\ vector \ = \hat{k}$ which is the correct orientation.