MA 200 - Lecture 15

1 Recap

1. Lebesgue's theorem
2. Fubini's theorem and evaluation of integrals.

2 Integration over a bounded set

Let $S \subset \mathbb{R}^n$ be a bounded set and $f : S \to \mathbb{R}$ be a bounded function. We define the characteristic function $\chi_S : \mathbb{R}^n \to \mathbb{R}$ of S as $\chi_S(x) = 1$ if $x \in S$ and 0 if $x \in S^c$. Then we say that f is Riemann integrable over S with integral $\int_S f dV$ if $f\chi_S$ is Riemann integrable over any rectangle Q containing S and define $\int_S f dV = \int_Q f\chi_S dV$.

Lemma 2.1. *Let $Q, Q' \subset \mathbb{R}^n$ be two rectangles. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded function that vanishes outside $Q \cap Q'$, then $\int_Q f dV$ exists iff $\int_{Q'} f dV$ does and they are both equal to each other.*

Proof. We shall prove that $\int_Q f = \int_{Q\cap Q'} f$
Suppose $\int_Q f dV$ exists: Choose P (and refine it so that the vertices of $Q \cap Q'$ belong to it and refine it even more by adding all points on edges at a distance of ϵ from the vertices) so that $U(P, f) - L(P, f) < \epsilon$. The partition P induces a partition P' of $Q \cap Q'$. Thus, $U(P', f) - L(P', f) < \epsilon + 2C\epsilon$ because $U(P', f)$ differs from $U(P, f)$ by at most $C\epsilon$. Hence $\int_{Q\cap Q'} f dV$ exists and equals $\int_Q f dV$.
Suppose $\int_{Q\cap Q'} f dV$ exists: Choose P' and extend to a partition P of Q by adding vertices at a distance of ϵ on all sides of P'. The previous argument goes through. □

Let $f, g : S \subset \mathbb{R}^n \to \mathbb{R}$ be two bounded functions. If f, g are continuous at $a \in S$, then it is easy to see that $\max(f, g)$ and $\min(f, g)$ are so too. Likewise, if f, g are Riemann-integrable over S, using Lebesgue's theorem it is easy to see that $\max(f, g)$ and $\min(f, g)$ are Riemann integrable over S. The Riemann integral satisfies a number of familiar properties:

1. Linearity ($\int_S (af + bg)$ exists if $\int_S f$ and $\int_S g$ exist and equals their linear combination): We can assume WLOG that S is a rectangle. On it, by Lebesgue, we see that $af + bg$ is R.I. Assume first that $a, b \geq 0$. For any partition, by linearity of sums, $a\int f + b\int g$ lies between $L(P, af + bg)$ and $U(P, af + b)$. So does $\int_S (af + bg)$. By refining partitions, we are done in this case. If we prove that $-\int f = \int(-f)$, we are done in general. Indeed, a simple partition argument does the job.

2. Comparison (If $f \leq g$, then $\int f \leq \int g$) and estimation $|\int f| \leq \int |f|$): For rectangles it is easy. For $|f|$, note that $|f| = \max(f, -f)$.

3. Monotonicity (If $T \subset S$ and $f \geq 0$, then $\int_T f \leq \int_S f$ assuming they exist): Note that $f_T \leq f_S$ and hence by comparison we are done.

4. Additivity (If $S = S_1 \cup S_2$, and f is R.I over S_1, S_2, then it is so over $S = S_1 \cup S_2, T = S_1 \cap S_2$ and $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$: Note that if $f \geq 0$, then $f_S = \max(f_{S_1}, f_{S_2})$ and $f_T = \min(f_{S_1}, f_{S_2})$. So these are R.I. If f is not non-negative, then $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ are R.I and moreover $f = f_+ - f_-$. So we are done. The additivity formula follows from $f_S = f_{S_1} + f_{S_2} - f_T$.

As a corollary, if $S_i \cap S_i$ has measure zero for all $i \neq j$, then $\int_S f = \int_{S_1} f + \int_{S_2} f + \ldots$. When is a continuous function f R.I over a bounded set S? The answer is provided by the following theorem.

Theorem 1. *Let $S \subset \mathbb{R}^n$ be a bounded set and $f : S \rightarrow \mathbb{R}$ be a bounded continuous function. Let $E \subset Bd(S)$ be the set of points x_0 such that $\lim_{x \to x_0} f(x) = 0$ fails to hold. Then if E has measure 0, f is R.I over S. (In particular, if $Bd(S)$ has measure zero - such domains are called rectifiable, a continuous bounded function is R.I over S.)*

Proof. Note that if $x \in E^c$, then either $x \in Int(S)$ (in which case x is a point of continuity of f) or $x \in Bd(S)$ but $\lim_{x \to x_0} f(x) = 0$ where x approaches from points in S. Now $f_S(x_0) = 0$ if $x_0 \in S$ (by continuity) or if x_0 is outside S by definition. Hence, for such points, $|f(x)| < \epsilon$ whenever $|x - x_0| < \delta$ and $x \in S$ (by assumption) or $x \in S^c$ (by definition). □

Of course, coming up with rectifiable domains does not appear to be trivial. Fortunately, your HW gives you a way to do so (the unit disc for instance). Lastly, Lebesgue's theorem and similar reasoning as above shows that

Theorem 2. *If $f : S \rightarrow \mathbb{R}$ is bounded continuous (and S bounded), then if f is R.I over S, it is R.I over $Int(S)$ and the integrals are equal.*