MA 200 - Lecture 16

1 Recap

1. Integration over a bounded set.

2. Properties of Riemann integrals (linearity, monotonicity, comparison, additivity).

2 Improper integrals

How can we define say $\int_0^1 \frac{1}{\sqrt{x}} dx$ or $\int_1^\infty \frac{1}{x^2} dx$? The easiest way is through limits. More generally,
Def: Let $A \subset \mathbb{R}^n$ be an open set (possibly unbounded) and $f : A \to \mathbb{R}$ be a continuous (not necessarily bounded) function. If $f \geq 0$ on A, we define the improper integral of f over A to be $\sup \int_D f$ where D ranges over compact rectifiable subsets of A provided this supremum is finite. In this case, we say that f is R.I over A in the improper sense. More generally, we say that f is R.I over A in the improper sense if f_+ and f_- are, and the improper integral is improper integral of f_+ minus that of f_-.
Are there enough compact rectifiable subsets at all in the first place? Thankfully yes:

Lemma 2.1. *Let $A \subset \mathbb{R}^n$ be open. Then there is a sequence C_N of compact rectifiable subsets such that $C_N \subset Int(C_{N+1})$ and $A = \cup_N C_N$. (Such a sequence is called a compact rectifiable exhaustion of A.)*

Proof. Let S_N be the set of all points in A that are at a distance of at least $\frac{1}{N}$ from the boundary of A (which is a closed set) and of distance at most N from the origin. Note that $A = \cup_N S_N$ (why?) and $S_N \subset Int(S_{N+1})$ (why?) Unfortunately, S_N need not be rectifiable. So we construct C_N by simply covering S_N by closed cubes that lie in the interior of C_{N+1}. We need only finitely many such cubes and their union is C_N. The boundary of C_N contains finitely many sets of measure zero and hence is of measure zero. Since $S_N \subset C_N$, the other properties are met. □

The following theorem can be proven by dividing into two cases ($f \geq 0$ and $f = f_+ - f_-$) and following one's nose.

Theorem 1. *Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ be continuous. Choose a compact rectifiable exhaustion C_N of A. Then f is improper R.I over A iff $\int_{C_N} |f|$ is bounded independent of N. In this case, $(Improper)\ \int_A f = \lim_{N\to\infty} \int_{C_N} f$.*

By passing to compact rectifiable exhaustions, one can prove linearity, comparison and estimation, monotonicity (this is a bit more subtle. In fact if $B \subset A$ is open, then $\int_B f$ exists and is less than or equal to $\int_A f$), and additivity.
If A is bounded, and f is bounded and continuous, does the improper integral coincide with the usual one? Thankfully, yes (if both integrals exist):

Theorem 2. *Let A be bounded and open, and $f : A \to \mathbb{R}$ be bounded and continuous. Then the improper integral exists. Suppose f is* also *R.I (this need not always happen). Then these integrals are equal.*

Proof. Since f is bounded and so is A, of course $\int_{C_N} |f|$ is bounded independent of N. Thus the improper integral exists. Now assume f is R.I. WLog $f \geq 0$. Indeed, if not, $f = f_+ - f_-$ where f_+, f_- are improper integrable and R.I (the former by definition and the latter by a theorem). Now by additivity of the usual and the improper integrals we reduce to the case where $f \geq 0$.
By monotonicity, if $D \subset A$ is compact and rectifiable, then $\int_D f \leq \int_A f$ in the usual sense. Since D is arbitrary, the improper integral is bounded above by the usual one.
Now let P be a partition of Q (containing A) and R_i be the subrectangles lying in A. Then if $D = \cup_i R_i$, we see that by properties (and the fact that $m_R(f_A) = m_R(f)$ if R is contained in A and $m_R(f_A) = 0 \leq m_R(f)$ otherwise), $L(f_A, P) \leq \int_D f \leq (Improper) \int_A f$. Since P is arbitrary, we are done. □

To apply these results for calculation, we need one more useful little theorem (it is annoying to choose a rectifiable compact exhaustion - it is easier to choose a rectifiable increasing sequence of open sets).

Theorem 3. *Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ be continuous. Let $U_1 \subset U_2 \ldots$ be open sets whose union is A. Then the improper integral exists iff $\int_{U_N} |f|$ is an existent bounded sequence. Its limit is the improper integral.*

Here is an example: Let $A = \{x > 1 \; and \; y > 1$ and $f(x, y) = \frac{1}{x^2 y^2}$. Now choose $U_N = (1, N) \times (1, N)$. Note that U_N is rectifiable and f being continuous on U_N is R.I and by Fubini and FTC, $\int_N f = (1 - 1/N)^2$ whose limit is 1.