MA 200 - Lecture 24

1 Recap

1. Defined tensors, symmetric tensors, alternating tensors, tensor products.
2. Proved that the dim of $\Lambda^k(V)$ was $\binom{n}{k}$ by introducing the basis ϵ_I.
3. Tried a naive definition of the wedge product.

2 Wedge product (Spivak's book)

Is the naive definition ($\epsilon_I \wedge \epsilon_J = \epsilon_{IJ}$ extended linearly) well-defined? That is, is it independent of the basis chosen? Yes. But this is rather painful to deal with. Nonetheless, assuming it is well-defined, here is a bunch of properties it satisfies.

1. $f \wedge (g \wedge h) = (f \wedge g) \wedge h$: Denote by I_{inc} the increasing order version of the multiindex I. $f \wedge (\sum_{JK} g_J h_K \epsilon_{JK})$. Now $\epsilon_{JK} = \epsilon_{(JK)_{inc}} sgn(\sigma_{JK \to (JK)_{inc}})$ and $f \wedge (g \wedge h) = \sum_{I,J,K} f_I g_J h_K \epsilon_{I(JK)_{inc}} sgn(\sigma_{JK \to (JK)_{inc}}) = \sum f_I g_J h_K \epsilon_{IJK}$. Likewise, $(f \wedge g) \wedge h$ is also given by the same expression.
2. $f \wedge g = (-1)^{kl} g \wedge f$ (So in particular, if f is a 1-form, $f \wedge f = 0$. Not necessarily true if f is a 2-form!): $f \wedge g = \sum f_I g_J \epsilon_{IJ} = \sum g_J f_I \epsilon_{JI} sgn(IJ \to JI)$. Now to take IJ to JI, we need to "slide" i_k past l, $j's$ and hence pick up $(-1)^l$. Likewise, for i_{k-1} and so on. Thus we get $(-1)^{kl}$.
3. If I is an increasing multi-index, then $\epsilon_I = \epsilon_{i_1} \wedge \epsilon_{i_2} \wedge \ldots$ (by induction and the first property). In fact, by the previous property, this is true for non-increasing multi-indices too.
4. $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$: Easy.
5. $(f + g) \wedge h = f \wedge h + g \wedge h$ and $f \wedge (g + h) = f \wedge g + f \wedge h$: Easy.

Suppose $T : V \to W$ is a linear map, then the linear map $T^* : \Lambda^k W \to \Lambda^k V$ is defined as $T^*(S)(v_1, \ldots, v_k) = S(Tv_1, \ldots, Tv_k)$ for $k \geq 1$. We shall now define a wedge product that satisfies the above properties and $T^*(f \wedge g) = T^* f \wedge T^* g$. Let us prove the existence of the wedge product (satisfying the properties we want). Firstly, taking cue from $2A = (A + A^T) + (A - A^T)$, we define the following operation:
Let T be a k-tensor on V. Then $Alt(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma(1)}, \ldots)$.
So if T is a 2-tensor, then $Alt(T)(v, w) = \frac{1}{2}(T(v, w) - T(w, v))$, that is, our familiar antisymmetrisation operation. In general, we have the following result.

Theorem 1. *1. $Alt(T) \in \Lambda^k(V)$*

2. If ω is a k-form, then $Alt(\omega) = \omega$

3. $Alt(Alt(T)) = Alt(T)$

Proof. 1. $k!Alt(T)(v_{\tau(1)}, \ldots) = \sum_\sigma sgn(\sigma)T(v_{\tau(\sigma(1))}, \ldots) = sgn(\tau)\sum_\sigma sgn(\sigma' = \tau \circ \sigma)T(v_{\sigma'(1)}, \ldots)$. But $\sigma \to \sigma'$ is an isomorphism and hence the summation over σ is the same as the summation over σ'. Hence we are done.

2. $Alt(\omega)(v_1, \ldots) = \frac{1}{k!}\sum_\sigma sgn(\sigma)\omega(v_{\sigma(1)}, \ldots) = \frac{1}{k!}\sum_\sigma sgn(\sigma)sgn(\sigma)\omega(v_1, \ldots) = \omega(v_1, \ldots)$.

3. Trivially from the first two properties.

□

Def: We define the wedge product $\omega \wedge \eta$ as $\frac{(k+l)!}{k!l!}Alt(\omega \otimes \eta)$.
It satisfies the following properties:

1. Bilinearity in f, g: Easy (because the tensor product is so).

2. $T^*(\omega \wedge \eta) = T^*\omega \wedge T^*\eta$: $\frac{(k+l)!}{k!l!}\sum sgn(\sigma)\omega \otimes \eta(Tv_{\sigma(1)}, \ldots) = \frac{(k+l)!}{k!l!}\sum sgn(\sigma)\omega(Tv_{\sigma(1)}, \ldots)\eta(Tv_{\sigma(k+1)}, \ldots$ $T^*\omega \wedge T^*\eta$.

3. $f \wedge (g \wedge h) = (f \wedge g) \wedge h$.

4. $f \wedge g = (-1)^{kl} g \wedge h$.

5. $\epsilon_I = \epsilon_{i_1} \wedge \ldots$: The properties above follow from the observation that $\omega \wedge \eta = \sum \omega_I \eta_J \epsilon_I \wedge \epsilon_J$ and the fact that $\epsilon_I \wedge \epsilon_J = \epsilon_{IJ}$: $\epsilon_I \wedge \epsilon_J(e_{\alpha_1}, \ldots) = 0$ if $\alpha_1, \ldots$ is *not* a permutation of IJ (why?). If it is a permutation, then $\epsilon_I \wedge \epsilon_J(e_{\alpha_1}, \ldots) = \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}} sgn(\sigma)\epsilon_I(e_{\alpha_{\sigma(1)}}, \ldots)\epsilon_J(e_{\alpha_{\sigma(k+1)}}, \ldots)$. Since we know that the wedge product produces $k + l$ forms, the α_i are all distinct (otherwise we will get zero anyway). Likewise, we can assume WLOG that $(\alpha_1, \ldots, \alpha_k) = I$ and $(\alpha_{k+1}, \ldots) = J$. Moreover, only those permutations survive in the summation that are of the form $\sigma = \sigma_1\sigma_2$ where σ_1 permutes only the I indices and σ_2 only the J-indices (in particular, if I and J have indices in common, $\epsilon_I \wedge \epsilon_J$ is 0). Thus, $\epsilon_I \wedge \epsilon_J(e_{\alpha_1}, \ldots) = \frac{1}{k!l!}\sum_{\sigma_1, \sigma_2} 1 = 1$ which is exactly $\epsilon_{IJ}(e_{i_1}, \ldots, e_{j_1}, \ldots)$.

3 Form fields in $\mathbb{R}^n$

Just as there are vector fields on $\mathbb{R}^n$ (basically, nicely varying collections of vectors, one for each point, like the electric and magnetic fields - and not like the BS fields of "positive" and "negative" "energy" spouted by pseudoscientists), we can define tensor fields and form fields:
Def: A smooth k-tensor field T on a set $S \subset \mathbb{R}^n$ is simply a tensor for every point $x \in S$ such that $T(x) = \sum_I c_I(x)\phi_I$ where $c_I(x)$ are smooth functions on S. A k form field ω, or sometimes a differential k-form, or sometimes, fondly (if you have nothing else to be fond of), a smooth k-form is simply a k-form for every point $x \in S$ such that $\omega(x) = \sum_{i_1 < i_2 < \ldots} \omega_I(x)\epsilon_I$ where $\omega_I(x)$ are smooth functions on S.

For instance, $\omega = x^2\epsilon_{12} + e^{yz}\epsilon_{23} + \sin(\sin(\cos(xzw)))\epsilon_{14}$ is a 2-form field on $\mathbb{R}^4$.
We can define the wedge product of these differential forms: $\omega \wedge \eta(x) := \omega(x) \wedge \eta(x)$.
When $k = 0$, we are talking about a number for every point x that varies smoothly, i.e., a smooth 0-form is simply a smooth function.