MA 200 - Lecture 11

1 Recap

1. Lagrange's multipliers. item Injective derivative theorem.
2. Definition of manifolds non-examples.

2 Manifolds in $\mathbb{R}^n$

Examples and non-examples:

1. The graph of $y = |x|$ is not an example of a C^1 manifold in $\mathbb{R}^2$ because if it were, then near the origin, there is a coordinate parametrisation $\alpha(t) = (x(t), y(t))$ such that $y(t) = |x(t)|$ and $\alpha(0) = (0, 0)$. Thus $y^2 = x^2$ and hence $yy' = xx'$. Since it is a coordinate parametrisation, either $x' \neq 0$ and therefore, y' is not continuous at $t = 0$ (why?) or $y' \neq 0$ and hence x' is not continuous at $t = 0$.
2. The circle is an example of a 1-dimensional smooth manifold (why?). Note that the circle cannot be covered by a *single* coordinate parametrisation (because the image of a coordinate parametrisation is not compact whereas a circle is).
3. Any regular level set is an example of an $n - 1$-dimensional smooth manifold (why?).

In other words, if f attains an extremum on a manifold-without-boundary, and if α is a coordinate parametrisation, then $f \circ \alpha$ attains an *unconstrained* local extremum and hence its gradient is zero. This is the real point of Lagrange's multipliers.

We will return to manifolds much later (because they are the right objects for generalising the fundamental theorem of calculus to higher dimensions).

3 Taylor's theorem and the second derivative test

So far, we have only seen how to calculate global extrema but have no means of recognising whether a local extremum is a local maximum or a local minimum. In one-variable calculus, we have the famous second-derivative test (the following statement can be strengthened significantly, but we don't need to do so for our purposes):

Theorem 1. *Let $I \subset \mathbb{R}$ be open, and $f : I \to \mathbb{R}$ be a function that is C^2 in a neighbourhood of a. If $f'(a) = 0$ and $f''(a) < 0$, a is a local maximum (and likewise for local minima). Conversely, if a is a local maximum, then $f'(a) = 0$ and $f''(a) \leq 0$.*

We have already proven that if a is a local extremum, $f'(a) = 0$ (even in higher dimensions). To get further information, we need to approximate f better (than the linear approximation). To this end, we have Taylor's theorem in one-variable:

Theorem 2. *Let $U \subset \mathbb{R}$ be an open set and $f : U \to \mathbb{R}$ a C^k function on U. Let $a \in U$ and $|h| < \epsilon$ such that $(a-\epsilon, a+\epsilon) \in U$. Then the polynomial $p_{k,a}(h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2!} + \ldots + \frac{f^{(k)}(a)h^k}{k!}$ is the unique polynomial of degree $\leq k$ such that $\lim_{h\to 0} \frac{f(a+h)-p_{k,a}(h)}{h^k} = 0$. Moreover, if f is C^{k+1}, then $f(a+h) = p_{k,a}(h) + \frac{f^{(k+1)}(\eta)h^{k+1}}{(k+1)!}$, where η lies between a and $a+h$.*

Proof. We prove uniqueness first. Suppose p_1, p_2 are two such polynomials of degree $\leq k$. Then $\frac{p_1-p_2}{h^d}$ goes to 0 as $h \to 0$ for all $d \leq k$. Assume that the first non-zero coefficient of $p_1 - p_2$ is that of h^d. Then we get a contradiction.
Now let $g(h) = f(a+h) - p_{k,a}(h)$. Note that g is C^k on a neighbourhood of 0, and $g^{(i)}(0) = 0$ for all $0 \leq i \leq k$ (why?) For $k = 1$, we are done easily by definition of the derivative. Assume Taylor's theorem for $1, 2 \ldots, k-1$. We apply this induction hypothesis to $g'(h)$. Hence, $\frac{g'(h)}{h^{k-1}} \to 0$. Now $g(h) = g'(\zeta_h)h$ (by LMVT) and hence $\frac{g(h)}{h^k} = \frac{g'(\zeta_h)}{\zeta_h^{k-1}} \frac{\zeta_h^{k-1}}{h^{k-1}}$ which goes to 0 by the squeeze rule.
Now we prove the remainder formula. For $k = 0$ it is easy (by LMVT). Hence, assume the truth of this statement for $0, 1, 2 \ldots, k-1$. For k, apply the induction hypothesis to $g'(h)$ to conclude that $g'(t) = \frac{g^{(k+1)}(\zeta_t)}{k!} t^k$. Considering $g(t)$ and t^{k+1} and using Cauchy's Mean Value Theorem, we see that $g'(c)h^{k+1} = (k+1)c^k g(h)$. Hence $g(h) = \frac{g^{k+1}(\theta)h^{k+1}}{(k+1)!} = \frac{f^{k+1}(\theta_h)h^{k+1}}{(k+1)!}$. □

Actually, Taylor's theorem holds even without the assumption of being C^k. In fact, k-times differentiable is good enough. But to prove such a thing, we need to use L'Hopital's rule.
Here is the proof of the second-derivative test: By Taylor, $f(a+h) = f(a) + f'(a)h + \frac{f''(\zeta)h^2}{2!}$ which means that $f(a+h) - f(a) = \frac{f''(\zeta_h)h^2}{2!}$. Since f is C^2, if $f''(a) > 0$, then in neighbourhood of a, $f''(x) > 0$. Thus, $f(a+h) - f(a) > 0$ in a neighbourhood of $h = 0$ and hence we are done. □

4 Taylor's theorem and the second derivative test

To state Taylor's theorem in multivariable calculus, we need some notation. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a "multi-index". We typically denote as follows: $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, $\alpha! = \alpha_1!\alpha_2!\ldots$, $h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \ldots$, and $D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \ldots f$ (note that the order does not matter thanks to Clairaut if f is $C^{|\alpha|}$).

Theorem 3. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ be a C^k function on U. Let $a \in U$ and $|h| < \epsilon$ such that $B_a(\epsilon) \subset U$. Then the polynomial $p_{a,k}(h) = f(a) + \sum_i \frac{\partial f}{\partial x_i}(a)h +$*

$\frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) h_i h_j + \ldots + \sum_{\|\alpha\|=k} \frac{D^\alpha f(a)}{\alpha!} h^\alpha$ *is the unique polynomial of degree* $\leq k$ *(degree meaning the maximum sum of powers) such that* $\lim_{h\to 0} \frac{f(a+h)-p_{k,a}(h)}{|h|^k} = 0$*. Moreover, if* f *is* C^{k+1}*, then* $f(a+h) = p_{k,a}(h) + \sum_{|\alpha|=k+1} \frac{D^\alpha f(\eta) h^\alpha}{\alpha!}$*, where* η *lies in* $B_a(h)$*.*

Proof. Uniqueness will be left as a HW problem.

Let $h \neq 0$ (if it is equal to 0, we are done). Consider the one-variable function $q(t) = f(a + t\frac{h}{\|h\|})$ on $|t| < \epsilon$. This function is C^k (because it is a composition of C^k functions). Thus we can apply the one-variable Taylor theorem to it to conclude that $q(\|h\|) = q(0) + q'(0)\|h\| + \ldots$.

Now we claim inductively that $\frac{q^{(m)}(t)\|h\|^m}{m!} = \sum_{|\alpha|=m} \frac{D^\alpha f(a+t\frac{h}{\|h\|})h^\alpha}{\alpha!}$:

Indeed, for $m = 1$ we are done by the Chain rule. Assume the truth of this statement for $1, 2\ldots, m-1$. We apply the induction hypothesis to $q^{(m-1)}(t)$ to conclude that $\frac{q^{(m)}(t)\|h\|^m}{m!} = \frac{\|h\|}{m}\sum_{|\alpha|=m-1} \frac{d}{dt}\frac{D^\alpha f(a+t\frac{h}{\|h\|})h^\alpha}{\alpha!} = \sum_{|\alpha|=m-1}\sum_i \frac{\partial_{x_i} D^\alpha f(a+t\frac{h}{\|h\|})h_i h^\alpha}{\alpha! m} = \sum_i \sum_{|\alpha|=m-1} \frac{\partial_{x_i} D^\alpha f(a+t\frac{h}{\|h\|})h_i h^\alpha}{\alpha! m}$. We want to compare the last expression to $\sum_{|\beta|=m} \frac{D^\beta f(a+t\frac{h}{\|h\|})h^\beta}{\beta!}$.

To be continued..... $\square$