MA 200 - Lecture 25

1 Recap

1. Defined the wedge product using Alt and defined form fields in $\mathbb{R}^n$.

2 Exterior derivative and pullback

We now want to generalise curl. Before that, we speak of the gradient in terms of forms:
Def: Let $U \subset \mathbb{R}^k$ or $U \subset \mathbb{H}^k$ be an open set and $f : U \to \mathbb{R}$ be a smooth 0-form, i.e., a smooth function. Then $df : U \to \Lambda^1$ is a smooth 1-form defined as $df(v) = \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_2}v_2 + \ldots$.
With this definition, note that $dx_1(v) = v_1$, $dx_2(v) = v_2$ and so on, i.e., $\epsilon_i = dx_i$ (if $e_1, \ldots$ is the standard basis), $\epsilon_I = dx_{i_1} \wedge dx_{i_2} \ldots$. Also, $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. This is one way of trying to make Newton's "infinitesimals" in early calculus rigorous. (Indeed, if you move infinitesimally for a time dt in a direction v, then f is expected to change by $\sum_i \frac{\partial f}{\partial x_i} v_i dt$ and moreover, $df \wedge df = 0$ ("second order"). But be careful! $dx_i \wedge dx_j \neq 0$. The correct way to make sense of infinitesimals is through ultrafilters (non-standard analysis) and is very complicated.)
More generally, we want to define the "curl" of ω where ω is a smooth k-form on U. We can naively try the same algorithm as the usual curl ($\nabla \times \vec{F}$), i.e., "$d \times \omega = (\frac{\partial}{\partial x_1} dx_1 + \ldots) \wedge \omega = \sum_{I_{inc}, j} \frac{\partial \omega_I}{\partial x_j} dx_j \wedge dx_I = \sum_I d\omega_I \wedge dx_I$". Indeed, we have the following theorem:

Theorem 1. *Let $U \subset \mathbb{R}^k$ or $\mathbb{H}^k$ be open. Denote by $\Omega^k(U)$ the infinite-dimensional vector space of smooth k-form fields on U. Then there exists a unique linear map $d : \Omega^k(U) \to \Omega^{k+1}(U)$ called the exterior derivative satisfying the following properties.*

1. *If f is a smooth 0-form on U, then $df(x) = \frac{\partial f}{\partial x_1}v_1 + \ldots$.*
2. *If ω, η and k, l-forms respectively, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ (that is, you almost pretend that d is a 1-form, just as you pretend that ∇ is a vector).*
3. *$d(d\omega) = 0$ for all ω (the analogue of $\nabla.(\nabla \times \vec{F}) = \vec{0}$).*

Proof. Define $d\omega$ as above by $d\omega := \sum_I d\omega_I \wedge \epsilon_I$ (and df as above for any function f). Clearly, this d is linear in ω. If ω, η are k, l forms, $d(\omega \wedge \eta) = d(\sum_{I_{inc}, J_{inc}} \omega_I \eta_J \epsilon_{IJ}) =$

$\sum(d\omega_I\eta_J \wedge \epsilon_{IJ} + \omega_I d\eta_J \wedge \epsilon_{IJ}) = d\omega \wedge \eta + \sum \omega_I (-1)^k \epsilon_I \wedge d\eta_J \wedge \epsilon_J = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. Lastly, $d(d\omega) = d(\sum_{i_1<\ldots} d\omega_I \wedge \epsilon_I) = \sum d(d\omega_I) \wedge \epsilon_I - d\omega_I d(\epsilon_I) = \sum d(d\omega_I) \wedge \epsilon_I$. Now $d(df) = \sum_i d(\frac{\partial f}{\partial x_i}\epsilon_i) = \frac{\partial^2 f}{\partial x_j \partial x_i}\epsilon_j \wedge \epsilon_i = 0$ because of Clairaut.
Now we prove uniqueness: Suppose $\omega = \sum_{I_{inc}} \omega_I \epsilon_I$. Then $d\omega$ by linearity is $\sum d(\omega_I \epsilon_I)$ which by the second property is $\sum d\omega_I \wedge \epsilon_I + \omega_I d(\epsilon_I)$. Now $\epsilon_i = dx_i$ by the first property. Moreover, $d(dx_i) = 0$ by the third property. Hence, by induction and the second property, $d(\epsilon_I) = 0$ and we are done. □

Examples:

1. $d(x^3 + y^2) = 3x^2 dx + 2y dy$.

2. $d(x^2 y dx \wedge dy + yw dz \wedge dx) \wedge (e^{yx} dz + \sin(xw) dw) = (0 + w dy \wedge dz \wedge dx + y dw \wedge dz \wedge dx) \wedge (e^{yx} dz + \sin(xw) dw)$ which is $w \sin(xw) dy \wedge dz \wedge dx \wedge dw$.

3. Let $\vec{F} = (P, Q, R)$. Consider the 1-form $\omega = P dx + Q dy + R dz$. Then $d\omega = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) dy \wedge dz + \ldots$ which corrresponds to $\nabla \times \vec{F}$.

4. $d(\frac{ydx - xdy}{x^2+y^2}) = 0$ but $\frac{ydx-xdy}{x^2+y^2} \neq df$ for any smooth function $f : \mathbb{R}^2 - \{(0,0\} \to \mathbb{R}$. In general, a form is said to be closed if $d\omega = 0$ and exact if $\omega = d\eta$. Clearly exact forms are closed but this example shows that the other way round is not true. This is true for certain kinds of sets. For instance, it is true if the domain of ω is all of $\mathbb{R}^n$. (How much this property fails tells us something about the shape of the domain. In fact, the quotient space of closed k-forms by exact k-forms is a useful object. It is called the k^{th} de Rham cohomology of the domain.)

Suppose you consider a form $\omega = x^2 dy + y^2 dx$ on $\mathbb{R}^2$. If $\gamma(t) = (\cos(t), \sin(t))$, we'd ideally like to say that "along γ, $dx = -\sin(t)dt$, $dy = \cos(t)dt$, $\omega = (\cos^3(t) - \sin^3(t))dt$" but the problem is that ω, dx, dy are form fields on $\mathbb{R}^2$ whereas dt is a form field on $\mathbb{R}$! Even pointwise, ω, dx, dy at a point $\gamma(t_0) = p$ are in $\Lambda^1(\mathbb{R}^2)$ whereas dt at a point t_0 is in $\Lambda^1(\mathbb{R})$. However, given a γ, we have a standard *linear* map that takes $\mathbb{R}$ to $\mathbb{R}^2$, i.e., $v \to D\gamma(v) = \langle \gamma'(t), v \rangle$. We can use the T^* construction now to make a definition:
Def: Suppose $F : U \subset \mathbb{R}^k$ or $\mathbb{H}^k$ to $\mathbb{R}^n$ is a smooth map, and ω is a smooth $l \geq 1$-form on $V \subset \mathbb{R}^n$ or $\mathbb{H}^n$ such that $F(U) \subset V$. Then the *pullback* $F^*\omega$ is a smooth l-form on U defined as $F^*\omega$ at the point x is simply $DF^*\omega$ at the point $F(x)$, i.e., $(F^*\omega)(x)(v_1, \ldots, v_l) = \omega(f(x))(Dfv_1, Dfv_2, \ldots)$. For 0-forms, $F^*f(x) = f \circ F(x)$.
Since we have already proven properties of T^*, we see that $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$ (this is true even if 0-form (why?)), F^* is linear, and $(G \circ F)^*\omega = F^* \circ G^*\omega$. Here are examples of pullbacks:

1. $\omega = dx \wedge dy$. Let $F(r, \theta) = (r\cos(\theta), r\sin(\theta))$. Then $F^*\omega = F^*dx \wedge F^*dy$. Now $F^*dx(v, w) = dx(DF(v, w)) = dx(\frac{\partial r\cos(\theta)}{\partial r} v + \frac{\partial r\cos(\theta)}{\partial \theta} w, \ldots) = \cos(\theta)v - r\sin(\theta)w = d(r\cos(\theta))(v, w)$. To be continued...