HW 1

1. Two norms $\|.\|_1, \|.\|_2$ on a vector space V are said to be equivalent if there exist positive constants c, C such that $c\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1, \ \forall \ v \in V$. Prove that any two norms on a finite-dimensional vector space are equivalent.

2. Prove that $\|AB\| \leq \|A\|\|B\|$ where the norms are either Frobenius or Operator, and A is an $m \times n$ real matrix and B is an $n \times p$ real matrix. Give examples where the inequality is strict.

3. Suppose A is a real symmetric $n \times n$ matrix. Prove that $\|A\|_{Operator} = \max_i |\lambda_i|$ where λ_i are the eigenvalues.

4. Let $f(x, y) = x^2 \sin(y/x)$ when $x \neq 0$ and 0 when $x = 0$. Prove that f is continuous everywhere.

5. Give an example of a set $S \subset \mathbb{R}^2$ and a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous at $a \in S$ when restricted to S but is not continuous at a when considered as a function on $\mathbb{R}^2$.

6. Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a function and $a \in U$ is not necessarily an interior point. We want to define f to be differentiable at a with derivative $L : \mathbb{R}^n \to \mathbb{R}^m$ if there exists an open set V containing a and a function $g : V \to \mathbb{R}^m$ such that $g(x) = f(x) \ \forall \ x \in V \cap U$ and g is differentiable at a with derivative L.

   (a) By means of a counterexample, show that this definition is not well-defined in general, i.e., it could depend on the choice of g, V.

   (b) Suppose U is the closed unit ball centred at the origin, then show that for all $a \in U$, the definition makes sense, i.e., if there is one $g : V \to \mathbb{R}^m$ that is differentiable at a with derivative L, then any other $\tilde{g} : \tilde{V} \to \mathbb{R}^m$ that is differentiable at a has the same derivative.