HW 4

1. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be a C^1 function. Suppose Df_a is invertible (where $a \in U$). Prove that there is an $r > 0$ such that

   (a) $B_a(r) \subset U$.

   (b) Df_x is invertible on $B_a(r)$ and $\|[Df_x]^{-1}\| \leq C$ for all $x \in B_a(r)$.

   (c) Let $x_1, x_1 + h \in B_a(r)$. Define Δ by $Df_{x_1}^{-1}(f(x_1 + h) - f(x_1)) = h + Df_{x_1}^{-1}\Delta$. Now $\|h + Df_{x_1}^{-1}\Delta\| \geq \frac{\|h\|}{2}$.

2. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be a C^1 function. Suppose Df_a is invertible (where $a \in U$). Prove that there exists $r' > 0$ such that $B_a(r') \subset U$, and that if $y_1, y_2, \ldots, y_n, z_1, \ldots, z_n \in B_a(r')$ then $\|AB^{-1} - I\|_{Frob} < \frac{1}{2}$, where the i^{th} row of A is $\nabla f_i(y_i)$ and that of B is $\nabla f_i(z_i)$.

3. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (xy, x^2 + y^2 + e^{(x-2)(y-1)})$. Prove that there exists an $r > 0$ such that for every $(a, b) \in B_{(2,6)}(r)$, there exists an $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = (a, b)$.