HW 5

1. Let $F(x, y, z) : \mathbb{R}^3 \to \mathbb{R}^2$ be defined as $F(x, y, z) = (x^2 + y^2 + z^2, xyz)$. Prove that there exists a neighbourhood of $(\frac{21}{4}, 1)$ such that for every (α, β) in this neighbourhood, there exists (a, b, c) such that $F(a, b, c) = (\alpha, \beta)$.

2. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ (where U is open) be a C^1 function. Suppose $\nabla f(a) \neq 0$ and $f(a) = 0$. Prove or disprove (by means of a counterexample) that there exists a neighbourhood $V \subset U$ of a such that the tangent plane at a does not intersect the portion of the level set $f(x) = 0$ lying in this neighbourhood (other than at $x = a$).

3. Given $f : \mathbb{R}^5 \to \mathbb{R}^2$, of class C^1. Let $a = (1, 2, -1, 3, 0)$; suppose that $f(a) = 0$ and

$$Df(a) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}$$

   (a) Show that there is a function $g : B \to \mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}^3$, containing the point $(1, 3, 0)$ such that $f(x_1, g_1(x), g_2(x), x_2, x_3) = 0$ for $x = (x_1, x_2, x_3) \in B$ and $g(1, 3, 0) = (2, -1)$.

   (b) Find $Dg(1, 3, 0)$.