HW 8

1. Prove the following theorem: Let $f : Q \subset \mathbb{R}^n \to \mathbb{R}$ be a bounded function on a closed rectangle Q. Then f is Riemann integrable over Q iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition P whose mesh is $< \delta$, $U(P, f) - L(P, f) < \epsilon$.

2. Show that if $A \subset \mathbb{R}^n$ has measure 0, then $\bar{A}$ need not have measure zero.

3. Show that no open subset of $\mathbb{R}^n$ has measure 0 in $\mathbb{R}^n$.

4. Show that $\mathbb{R}^2 \times \{0\}$ has measure 0 in $\mathbb{R}^3$.

5. Let $S \subset \mathbb{R}^n$ be a compact set and let $f : S \to \mathbb{R}$ be a continuous function.

   (a) Prove that the graph $G_f \subset \mathbb{R}^{n+1}$ has measure 0.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^r function (where $r \geq 1$) be such that $S = f^{-1}(0)$ is not empty and $\nabla f(x) \neq 0 \ \forall\ x \in f^{-1}(0)$. Prove that $S \subset \mathbb{R}^n$ has measure 0 in $\mathbb{R}^n$.