HW 9

1. Let $S \in \mathbb{R}^{n+1}$ consist of (x, y) where $x \in Q \subset \mathbb{R}^n$ (where Q is a closed rectangle) and $y \in \mathbb{R}$ such that $\phi_1(x) \leq y \leq \phi_2(x)\ \forall\ x \in Q$ for two smooth functions $\phi_1(x), \phi_2(x) : \mathbb{R}^n \to \mathbb{R}$. Suppose $f : S \to \mathbb{R}$ is a continuous function. Prove that f is Riemann integrable and write its integral over S as an iterated integral (with proof of course).

2. Let $A \subset \mathbb{R}^2$ be defined by $A = \{(x, y) \mid x > 1\ \textit{and}\ 0 < y < 1/x\}$. Calculate $\int_A \frac{1}{x\sqrt{y}}$ if it exists.

3. In this exercise, you shall prove that if $S \subset \mathbb{R}^n$ is any set with measure zero in $\mathbb{R}^n$, and $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function (where $S \subset U$ and U is an open set), then $f(S) \subset \mathbb{R}^n$ has measure zero.

   (a) Let C_N be a compact rectifiable exhaustion of U and $S = \cup_N S_N$ where $S_N = S \cap C_N$. Prove that it is enough to show that S_N has measure 0 for all N. Now fix N for the rest of the exercise.

   (b) Given a rectangle Q and a number $\delta > 0$, prove that there is a finite cover of Q by rectangles of width $< \delta$ and total volume $< 2v(Q)$.

   (c) Prove that there exists $M > 0$ such that $\forall x, y \in C_{N+1}$, $\|f(x) - f(y)\| \leq M\|x - y\|$. As a consequence prove that if Q is a rectangle of width $< \delta$ in C_{N+1}, $f(Q)$ is contained in a rectangle Q' of width $< 2M\delta$ and $v(Q') < (2M)^n v(Q)$.

   (d) Prove that $f(S_N)$ has measure zero.