MA 200 - Lecture 25

1 Recap

1. Orientations for 1-manifolds through unit tangent vector fields.
2. Orientations for hypersurfaces through unit normal vector fields.
3. Restrictions of orientations from M to ∂M produce orientations for ∂M. A standard orientation for ∂M is the restriction if $dim(M)$ is even, and the opposite of the restriction when $dim(M)$ is odd.
4. Defined k-tensors, symmetric and alternating tensors.

2 Differential forms, wedge products, and form-fields in $\mathbb{R}^n$

Inner products are examples of symmetric 2-tensors and the determinant an example of $\Lambda^n(\mathbb{R}^n)$. In fact, we can produce alternating 3-tensors in $\mathbb{R}^4$ using the determinant as $(u, v, w) \to \det(e_1\ u\ v\ w)$ and so on. So what is the dimension of $\Lambda^k V$? Suppose $e_1, \ldots, e_n$ is a basis of V. Define the following alternating k-tensors for a given k-multiindex $I = (i_1, \ldots, i_k)$:

$$\epsilon_I(v_1, \ldots, v_k) = \det(A_I)\ where\ (A_I)_{jl} = (v_l)_{i_j}. \tag{1}$$

Here are examples:

1. When $k = 1$, ϵ_i form the dual basis for $V^* = \Lambda^1 V$.
2. Suppose $k = 2$ and $I = (2, 3)$, $v_1 = (a, b, c)$, $v_2 = (\alpha, \beta, \gamma)$ then $\epsilon_I(v_1, v_2) = \det \begin{pmatrix} b & \beta \\ c & \gamma \end{pmatrix} = b\gamma - c\beta$.
3. Suppose $i_1 = i_3$. Then $\epsilon_I = 0$ (why?) More generally, when two indices coincide, $\epsilon_I = 0$.
4. Recall that the sign of a permutation $\sigma \in S_k$ is simply the determinant of the corresponding permutation matrix. Now we define $\sigma(I) = (i_{\sigma(1)}, \ldots)$. (For instance, if $k = 3$, and $\sigma = (12)$, then $\sigma(2, 3, 1) = (3, 2, 1)$. Then $\epsilon_{\sigma(I)} = sgn(\sigma)\epsilon_I$ by the properties of determinants.

Now we consider the multiindices I such that $i_1 < i_2 < \ldots < i_k$. There are $\binom{n}{k}$ of these indices. We claim that such ϵ_I form a basis for $\Lambda^k(V)$. Indeed,

1. These ϵ_I are linearly independent: Suppose $\sum_{i_1<\ldots<i_k} c_I\epsilon_I = 0$. Then if $j_1 < j_2 < \ldots < j_k$, $0 = \sum c_I\epsilon_I(e_{j_1}, \ldots, e_{j_k}) = c_J$ (why?)

2. They span $\Lambda^k(V)$: Let $\omega \in \Lambda^k(V)$. Then consider $\tilde{\omega} = \sum \omega(e_{i_1}, \ldots, e_{i_k})\epsilon_I$. Note that to prove that $\tilde{\omega} = \omega$, it is enough to show that they are equal on $(e_{j_1}, \ldots, e_{j_k})$ for all increasing multiindices J (why? Because of multilinearity, and the facts that if two indices coincide, we get zero and if we change the ordering, we pick up the sign of the permutation). Now $\tilde{\omega}(e_J) = \sum \omega(e_I)\epsilon_I(e_J) = \omega(e_J)$ (where we abuse notation by denoting a tuple $(w_{i_1}, \ldots)$ by w_I).

Note that when $k = n$, Λ^n has dimension 1 and is generated by $\epsilon_{12\ldots n}$. These forms are also called 'top forms'. When $k = 0$, again the dimension is 1. For $n = 3$, Λ^1 and Λ^2 have exactly the same dimension equal to 3 (which is also the dimension of V!) So we can identify a vector $v \in \mathbb{R}^3$ with a 1-form $\omega = v_1\epsilon_1 + v_2\epsilon_2 + v_3\epsilon_3$ and with a 2-form $v_1\epsilon_{23} + v_2\epsilon_{31} + v_3\epsilon_{12}$ (why this weird identification? Because we want to think of ϵ_{12} as $\hat{i} \times \hat{j} = \hat{k}$).
We are now in a position to define the generalisation of the cross product. Instead of defining it for vectors, we define the wedge product $\wedge$ of forms. We want to try the following naive definition: Let $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$. Then $\omega \wedge \eta \in \Lambda^{k+l}(V)$ is defined as $\omega \wedge \eta = \sum_{i_1<\ldots<i_k, j_1<\ldots<j_l} \omega_I\eta_J\epsilon_{IJ}$, i.e., we define $\epsilon_I \wedge \epsilon_J$ as ϵ_{IJ} and extend this definition linearly.
Is this well-defined? That is, is it independent of the basis chosen? Yes. But this is rather painful to deal with. Nonetheless, assuming it is well-defined, here is a bunch of properties it satisfies.

1. $f\wedge(g\wedge h) = (f\wedge g)\wedge h$: Denote by I_{inc} the increasing order version of the multiindex I. $f \wedge (\sum_{J_{inc}K_{inc}} g_Jh_K\epsilon_{JK})$. Now $\epsilon_{JK} = \epsilon_{(JK)_{inc}}sgn(\sigma_{JK\to(JK)_{inc}})$ and $f \wedge (g \wedge h) = \sum_{I_{inc},J_{inc},K_{inc}} f_Ig_Jh_K\epsilon_{I(JK)_{inc}}sgn(\sigma_{JK\to(JK)_{inc}}) = \sum f_Ig_Jh_K\epsilon_{IJK}$. Likewise, $(f \wedge g) \wedge h$ is also given by the same expression.

2. $f \wedge g = (-1)^{kl}g \wedge f$ (So in particular, if f is a 1-form, $f \wedge f = 0$. Not necessarily true if f is a 2-form!): $f \wedge g = \sum f_Ig_J\epsilon_{IJ} = \sum g_Jf_I\epsilon_{JI}sgn(IJ \to JI)$. Now to take IJ to JI, we need to "slide" i_k past l, $j's$ and hence pick up $(-1)^l$. Likewise, for i_{k-1} and so on. Thus we get $(-1)^{kl}$.

3. If I is an increasing multi-index, then $\epsilon_I = \epsilon_{i_1} \wedge \epsilon_{i_2} \wedge \ldots$ (by induction and the first property). In fact, by the previous property, this is true for non-increasing multi-indices too.

4. $(cf) \wedge g = c(f \wedge g) = f \wedge (cg)$: Easy.

5. $(f + g) \wedge h = f \wedge h + g \wedge h$ and $f \wedge (g + h) = f \wedge g + f \wedge h$: Easy.

Suppose $T : V \to W$ is a linear map, then the linear map $T^* : \Lambda^kW \to \Lambda^kV$ is defined as $T^*(S)(v_1, \ldots, v_k) = S(Tv_1, \ldots, Tv_k)$ for $k \geq 1$. We shall in the next class define a wedge product that satisfies the above properties and $T^*(f \wedge g) = T^*f \wedge T^*g$.