# MA 200 - Lecture 18

## 1 Recap

1. Define measure zero sets and proved a few properties (about unions and so on).
2. Proved one part of Lebesgue's theorem (if $D$ has measure $0$...).

## 2 Measure zero sets and integrability (and everything else from Munkres)

**Theorem 1.** *Let $Q \subset \mathbb{R}^n$ be a closed rectangle and $f : Q \to \mathbb{R}$ be a bounded function. Let $D \subset Q$ be the set of discontinuities of $f$. Then $f$ is R.I iff $D$ has measure zero in $\mathbb{R}^n$.*

*Proof.* Let $|f(x)| \leq M \; \forall \, x \in Q$.

1. If $f$ is R.I: The idea is that if $U(P,f) - L(P,f) = \sum_R (M_R - m_R)v(R) < \epsilon$, then the rectangles that cover discontinuities ought to have a relatively large $M_R - v_R$. (If so, then their total volume is small and we are done.) This is not quite true (because $M_R - v_R$ can still be $O(\epsilon)$ for some such rectangles for instance). So we shall quantify the amount of discontinuity at every point:
   The oscillation of $f$ at a point $a \in Q$ is defined as $o(f,a) = \inf_{\delta>0}(M_\delta f - m_\delta f)$ where $M_\delta(f) = \sup_{|x-a|<\delta} f(x)$ and $m_\delta(f) = \sup_{|x-a|<\delta} f(x)$. It is easy to see that $o(f,a) = 0$ iff $f$ is continuous at $a$ (why?).
   Let $D_m$ be the set of all $x \in Q$ such that $o(f,x) \geq \frac{1}{m}$. Clearly, $D = \cup_{m \in \mathbb{Z}_{>0}} D_m$. If prove that each $D_m$ has measure zero, we are done. Indeed, since $f$ is R.I, given $\epsilon > 0$, there is a partition $P$ such that $\sum_R (M_R - m_R)v(R) < \frac{\epsilon}{2m}$. Now every point in $D_m$ is either in the boundary of a closed rectangle $R$ or in the interior of one. The collection of all the ones in the boundary have measure $0$ (because the boundaries of each of these countably many rectangles has measure $0$) and hence can be covered by countably many closed rectangles of total measure $\frac{\epsilon}{2}$. Suppose $Int(R_{i_1}), \ldots, Int(R_{i_k})$ are the rectangles containing all the other points of $D_m$. Then $\frac{1}{m}\sum_k v(R_{i_k}) \sum_k (M_{R_{i_k}} - m_(R_{i_k}))v(R_{i_k}) < \frac{\epsilon}{2m}$ and hence $\sum_k v(R_{i_k}) < \frac{\epsilon}{2}$. Thus we are done.

□

As a consequence, piecewise continuous functions (with finitely many pieces) are R.I. Moreover,

**Theorem 2.** *Assume $f$ is integrable over $Q$.*

1. *If $f = 0$ everywhere except on a set $E$ of measure $0$ (vanishes "almost everywhere"), then $\int_Q f = 0$.*

2. *If $f \geq 0$ and $\int_Q f = 0$, then $f$ vanishes almost everywhere.*

*Proof.*
1. Let $P$ be a partition. If $R$ is any subrectangle, $R$ is not contained in $E$ and hence has a point where $f$ vanishes. Thus, $m_R \leq 0$ and $M_R \geq 0$. Thus, the L.I is $\leq 0$ and the $U.I \geq 0$. Since they coincide, the integral is $0$.

2. Since the integral exists, $f$ is continuous almost everywhere. At all these points of continuity, we claim that $f = 0$. Indeed, if $f(a) > 0$ at some $a$, then $f(x) \geq c > 0$ for all $|x - a| < \delta$. Now, choose a partition $P$ of mesh $< \delta$. For any rectangle containing $a$, $m_R(f) \geq c$. Thus, $L(f, P) \geq v(R_0)c > 0$. But $L(f, P) \leq \int_Q f = 0$.

□

# 3 Evaluation of integrals

Recall the fundamental theorem of calculus.

**Theorem 3.** *If $f : [a, b] \to \mathbb{R}$ is continuous,*

1. *then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)\ \forall\ x \in [a, b]$.*

2. *and if $F(x)$ is an antiderivative of $f$ on $[a, b]$, i.e., $F'(x) = f(x)\ \forall\ x \in [a, b]$, then $F(b) - F(a) = \int_a^b f(x)dx$ .*

What about multiple integrals? How can we evaluate them?

**Theorem 4** (Fubini)**.** *Let $Q = A \times B$ where $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^n$ are rectangles. Let $f : Q \to \mathbb{R}$ be a bounded function. If $f$ is R.I over $Q$, then $\underline{\int_{y\in B}} f(x, y)dy$ and $\overline{\int_{y\in B}} f(x, y)dy$ are integrable over $A$ and $\int_Q f = \int_{x\in A} \underline{\int_{y\in B}} f(x, y)dydx = \int_{x\in A} \overline{\int_{y\in B}} f(x, y)dydx$. (In particular, if $f$ is continuous, then the iterated integral exists and equals the multiple integral in any order.)*

*Proof.* An easy comparison of sums over partitions (using of course, $v(R_A \times R_B) = v(R_A)v(R_B)$, and the finite-sum version of Fubini). □

Now we can integrate say, $x^2y + 4x^3z^2$ over a rectangle explicitly.

# 4 Integration over a bounded set

Let $S \subset \mathbb{R}^n$ be a bounded set and $f : S \to \mathbb{R}$ be a bounded function. We define the characteristic function $\chi_S : \mathbb{R}^n \to \mathbb{R}$ of $S$ as $\chi_S(x) = 1$ if $x \in S$ and $0$ if $x \in S^c$. Then we say that $f$ is Riemann integrable over $S$ with integral $\int_S f dV$ if $f\chi_S$ is Riemann integrable over any rectangle $Q$ containing $S$ and define $\int_S f dV = \int_Q f\chi_S dV$.

**Lemma 4.1.** *Let $Q, Q' \subset \mathbb{R}^n$ be two rectangles. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a bounded function that vanishes outside $Q \cap Q'$, then $\int_Q f dV$ exists iff $\int_{Q'} f dV$ does and they are both equal to each other.*

*Proof.* We shall prove that $\int_Q f = \int_{Q\cap Q'} f$
Suppose $\int_Q f dV$ exists: Choose $P$ (and refine it so that the vertices of $Q \cap Q'$ belong to it and refine it even more by adding all points on edges at a distance of $\epsilon$ from the vertices) so that $U(P, f) - L(P, f) < \epsilon$. The partition $P$ induces a partition $P'$ of $Q \cap Q'$. Thus, $U(P', f) - L(P', f) < \epsilon + 2C\epsilon$ because $U(P', f)$ differs from $U(P, f)$ by at most $C\epsilon$. Hence $\int_{Q\cap Q'} f dV$ exists and equals $\int_Q f dV$.
Suppose $\int_{Q\cap Q'} f dV$ exists: Choose $P'$ and extend to a partition $P$ of $Q$ by adding vertices at a distance of $\epsilon$ on all sides of $P'$. The previous argument goes through. □