MA 200 - Lecture 26

1 Recap

1. Proved that the dim of $\Lambda^k(V)$ was $\binom{n}{k}$ by introducing the basis ϵ_I.
2. Tried a naive definition of the wedge product (but failed to prove basis independence)! Nonetheless, we proved a bunch of properties assuming that the definition was well-defined. In fact, these properties uniquely determine the wedge product (if it exists) - this is easy to prove.
3. Defined $T^*\omega$. It is easy to show that $(G \circ F)^* = F^* \circ G^*$.

2 Wedge product (Spivak's book)

Let us prove the existence of the wedge product (satisfying the properties we want). Firstly, taking cue from $2A = (A + A^T) + (A - A^T)$, we define the following operation: Let T be a k-tensor on V. Then $Alt(T)(v_1, \ldots, v_k) = \frac{1}{k!}\sum_{\sigma \in S_k} sgn(\sigma)T(v_{\sigma(1)}, \ldots)$.
So if T is a 2-tensor, then $Alt(T)(v, w) = \frac{1}{2}(T(v, w) - T(w, v))$, that is, our familiar antisymmetrisation operation. In general, we have the following result.

Theorem 1. *1. $Alt(T) \in \Lambda^k(V)$*

2. If ω is a k-form, then $Alt(\omega) = \omega$

3. $Alt(Alt(T)) = Alt(T)$

Proof. 1. $k!Alt(T)(v_{\tau(1)}, \ldots) = \sum_\sigma sgn(\sigma)T(v_{\sigma(\tau(1))}, \ldots) = sgn(\tau)\sum_\sigma sgn(\sigma' = \sigma \circ \tau)T(v_{\sigma'(1)}, \ldots)$. But $\sigma \to \sigma'$ is an isomorphism and hence the summation over σ is the same as the summation over σ'. Hence we are done.

2. $Alt(\omega)(v_1, \ldots) = \frac{1}{k!}\sum_\sigma sgn(\sigma)\omega(v_{\sigma(1)}, \ldots) = \frac{1}{k!}\sum_\sigma sgn(\sigma)sgn(\sigma)\omega(v_1, \ldots) = \omega(v_1, \ldots)$.

3. Trivially from the first two properties.

□

Def: We define the wedge product $\omega \wedge \eta$ as $\frac{(k+l)!}{k!l!}Alt(\omega \otimes \eta)$.
It satisfies the following properties:

1. Bilinearity in f, g: Easy (because the tensor product is so).

2. $T^*(\omega\wedge\eta) = T^*\omega\wedge T^*\eta$: $\frac{(k+l)!}{k!l!}\sum sgn(\sigma)\omega\otimes\eta(Tv_{\sigma(1)},\ldots) = \frac{(k+l)!}{k!l!}\sum sgn(\sigma)\omega(Tv_{\sigma(1)},\ldots)\eta(Tv_{\sigma(k+1)},\ldots$ $T^*\omega \wedge T^*\eta$.

3. $f \wedge (g \wedge h) = (f \wedge g) \wedge h$.

4. $f \wedge g = (-1)^{kl} g \wedge h$.

5. $\epsilon_I = \epsilon_{i_1} \wedge \ldots$: The properties above follow from the observation that $\omega \wedge \eta = \sum \omega_I \eta_J \epsilon_I \wedge \epsilon_J$ and the fact that $\epsilon_I \wedge \epsilon_J = \epsilon_{IJ}$: $\epsilon_I \wedge \epsilon_J(e_{\alpha_1},\ldots) = 0$ if $\alpha_1,\ldots$ is *not* a permutation of IJ (why?). If it is a permutation, then $\epsilon_I \wedge \epsilon_J(e_{\alpha_1},\ldots) = \frac{1}{k!l!}\sum_{\sigma\in S_{k+l}} sgn(\sigma)\epsilon_I(e_{\alpha_{\sigma(1)}},\ldots)\epsilon_J(e_{\alpha_{\sigma(k+1)}},\ldots)$. Since we know that the wedge product produces $k+l$ forms, the α_i are all distinct (otherwise we will get zero anyway). Likewise, we can assume WLOG that $(\alpha_1,\ldots,\alpha_k) = I$ and $(\alpha_{k+1},\ldots) = J$. Moreover, only those permutations survive in the summation that are of the form $\sigma = \sigma_1\sigma_2$ where σ_1 permutes only the I indices and σ_2 only the J-indices (in particular, if I and J have indices in common, $\epsilon_I \wedge \epsilon_J$ is 0). Thus, $\epsilon_I \wedge \epsilon_J(e_{\alpha_1},\ldots) = \frac{1}{k!l!}\sum_{\sigma_1,\sigma_2} 1 = 1$ which is exactly $\epsilon_{IJ}(e_{i_1},\ldots,e_{j_1},\ldots)$.

3 Form fields in $\mathbb{R}^n$

Just as there are vector fields on $\mathbb{R}^n$ (basically, nicely varying collections of vectors, one for each point, like the electric and magnetic fields - and not like the BS fields of "positive" and "negative" "energy" spouted by pseudoscientists), we can define tensor fields and form fields:
Def: A smooth k-tensor field T on a set $S \subset \mathbb{R}^n$ is simply a tensor for every point $x \in S$ such that $T(x) = \sum_I c_I(x)\phi_I$ where $c_I(x)$ are smooth functions on S. A k form field ω, or sometimes a differential k-form, or sometimes, fondly (if you have nothing else to be fond of), a smooth k-form is simply a k-form for every point $x \in S$ such that $\omega(x) = \sum_{i_1<i_2<\ldots} \omega_I(x)\epsilon_I$ where $\omega_I(x)$ are smooth functions on S.
For instance, $\omega = x^2\epsilon_{12} + e^{yz}\epsilon_{23} + \sin(\sin(\cos(xzw)))\epsilon_{14}$ is a 2-form field on $\mathbb{R}^4$.
We can define the wedge product of these differential forms: $\omega \wedge \eta(x) := \omega(x) \wedge \eta(x)$.
When $k = 0$, we are talking about a number for every point x that varies smoothly, i.e., a smooth 0-form is simply a smooth function.

4 Exterior derivative

We now want to generalise curl. Before that, we speak of the gradient in terms of forms:
Def: Let $U \subset \mathbb{R}^k$ or $U \subset \mathbb{H}^k$ be an open set and $f : U \to \mathbb{R}$ be a smooth 0-form, i.e., a smooth function. Then $df : U \to \Lambda^1$ is a smooth 1-form defined as $df(v) = \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_2}v_2 + \ldots$.
With this definition, note that $dx_1(v) = v_1$, $dx_2(v) = v_2$ and so on, i.e., $\epsilon_i = dx_i$ (if $e_1,\ldots$ is the standard basis), $\epsilon_I = dx_{i_1} \wedge dx_{i_2} \ldots$. Also, $df = \sum_i \frac{\partial f}{\partial x_i}dx_i$. This is one way of trying to make Newton's "infinitesimals" in early calculus rigorous. (Indeed, if you

move infinitesimally for a time dt in a direction v, then f is expected to change by $\sum_i \frac{\partial f}{\partial x_i} v_i dt$ and moreover, $df \wedge df = 0$ ("second order"). But be careful! $dx_i \wedge dx_j \neq 0$. The correct way to make sense of infinitesimals is through ultrafilters (non-standard analysis) and is very complicated.)
More generally, we want to define the "curl" of ω where ω is a smooth k-form on U. We can naively try the same algorithm as the usual curl ($\nabla \times \vec{F}$), i.e., "$d \times \omega = (\frac{\partial}{\partial x_1} dx_1 + \ldots) \wedge \omega = \sum_{I_{inc}, j} \frac{\partial \omega_I}{\partial x_j} dx_j \wedge dx_I = \sum_I d\omega_I \wedge dx_I$". Indeed, we have the following theorem:

Theorem 2. *Let $U \subset \mathbb{R}^k$ or $\mathbb{H}^k$ be open. Denote by $\Omega^k(U)$ the infinite-dimensional vector space of smooth k-form fields on U. Then there exists a unique linear map $d : \Omega^k(U) \to \Omega^{k+1}(U)$ called the exterior derivative satisfying the following properties.*

1. *If f is a smooth 0-form on U, then $df(x) = \frac{\partial f}{\partial x_1} v_1 + \ldots$.*
2. *If ω, η and k, l-forms respectively, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ (that is, you almost pretend that d is a 1-form, just as you pretend that ∇ is a vector).*
3. *$d(d\omega) = 0$ for all ω (the analogue of $\nabla.(\nabla \times \vec{F}) = \vec{0}$).*

Proof. Define $d\omega$ as above by $d\omega := \sum_I d\omega_I \wedge \epsilon_I$ (and df as above for any function f). Clearly, this d is linear in ω. If ω, η are k, l forms, $d(\omega \wedge \eta) = d(\sum_{I_{inc}, J_{inc}} \omega_I \eta_J \epsilon_{IJ}) = \sum (d\omega_I \eta_J \wedge \epsilon_{IJ} + \omega_I d\eta_J \wedge \epsilon_{IJ}) = d\omega \wedge \eta + \sum \omega_I (-1)^k \epsilon_I \wedge d\eta_J \wedge \epsilon_J = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. Lastly, $d(d\omega) = d(\sum_{i_1 < \ldots} d\omega_I \wedge \epsilon_I) = \sum d(d\omega_I) \wedge \epsilon_I - d\omega_I d(\epsilon_I) = \sum d(d\omega_I) \wedge \epsilon_I$. Now $d(df) = \sum_i d(\frac{\partial f}{\partial x_i} \epsilon_i) = \frac{\partial^2 f}{\partial x_j \partial x_i} \epsilon_j \wedge \epsilon_i = 0$ because of Clairaut.
Now we prove uniqueness: Suppose $\omega = \sum_{I_{inc}} \omega_I \epsilon_I$. Then $d\omega$ by linearity is $\sum d(\omega_I \epsilon_I)$ which by the second property is $\sum d\omega_I \wedge \epsilon_I + \omega_I d(\epsilon_I)$. Now $\epsilon_i = dx_i$ by the first property. Moreover, $d(dx_i) = 0$ by the third property. Hence, by induction and the second property, $d(\epsilon_I) = 0$ and we are done. □

Examples:

1. $d(x^3 + y^2) = 3x^2 dx + 2y dy$.
2. $d(x^2 y dx \wedge dy + yw dz \wedge dx) \wedge (e^{yx} dz + \sin(xw) dw) = (0 + w dy \wedge dz \wedge dx + y dw \wedge dz \wedge dx) \wedge (e^{yx} dz + \sin(xw) dw)$ which is $w \sin(xw) dy \wedge dz \wedge dx \wedge dw$.
3. Let $\vec{F} = (P, Q, R)$. Consider the 1-form $\omega = P dx + Q dy + R dz$. Then $d\omega = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) dy \wedge dz + \ldots$ which corrresponds to $\nabla \times \vec{F}$.
4. $d(\frac{ydx - xdy}{x^2 + y^2}) = 0$ but $\frac{ydx - xdy}{x^2 + y^2} \neq df$ for any smooth function $f : \mathbb{R}^2 - \{(0, 0\} \to \mathbb{R}$. In general, a form is said to be closed if $d\omega = 0$ and exact if $\omega = d\eta$. Clearly exact forms are closed but this example shows that the other way round is not true. This is true for certain kinds of sets. For instance, it is true if the domain of ω is all of $\mathbb{R}^n$. (How much this property fails tells us something about the shape of the domain. In fact, the quotient space of closed k-forms by exact k-forms is a useful object. It is called the k^{th} de Rham cohomology of the domain.)

Suppose you consider a form $\omega = x^2dy + y^2dx$ on $\mathbb{R}^2$. If $\gamma(t) = (\cos(t), \sin(t))$, we'd ideally like to say that "along γ, $dx = -\sin(t)dt$, $dy = \cos(t)dt$, $\omega = (\cos^3(t) - \sin^3(t))dt$" but the problem is that ω, dx, dy are form fields on $\mathbb{R}^2$ whereas dt is a form field on $\mathbb{R}$! Even pointwise, ω, dx, dy at a point $\gamma(t_0) = p$ are in $\Lambda^1(\mathbb{R}^2)$ whereas dt at a point t_0 is in $\Lambda^1(\mathbb{R})$. However, given a γ, we have a standard *linear* map that takes $\mathbb{R}$ to $\mathbb{R}^2$, i.e., $v \to D\gamma(v) = \langle \gamma'(t), v \rangle$. We can use the T^* construction now to make a definition:
Def: Suppose $F : U \subset \mathbb{R}^k$ or $\mathbb{H}^k$ to $\mathbb{R}^n$ is a smooth map, and ω is a smooth $k \geq 1$-form on $V \subset \mathbb{R}^k$ or $\mathbb{H}^k$ such that $F(U) \subset V$. Then the *pullback* $F^*\omega$ is a smooth k-form on U defined as $F^*\omega$ at the point x is simply $DF^*\omega$ at the point $F(x)$, i.e., $(F^*\omega)(x)(v_1, \ldots, v_k) = \omega(f(x))(Dfv_1, Dfv_2, \ldots)$.