MA 200 - Lecture 7

1 Recap

1. Proved the chain rule.
2. Mean value theorem.
3. Defined C^r. By the way, if f is C^r on an open set U for all r, then it is said to be C^∞ (read as "smooth").

Theorem 1. *(Clairaut): If $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is C^2 on U (and U is open), then $D_i D_j f(a) = D_j D_i f(a)$ for all $i \neq j$.*

By the way, it is enough for f to merely be twice-differentiable (without being C^2) but the proof is more complicated.

Proof. Since we are interested in two variables at a time, without loss of generality, we can assume that $f(x, y)$ is a function of merely two variables.

We first prove a second-order mean value theorem. Let $\lambda(h, k) = f(a, b) - f(a+h, b) - f(a, b+k) + f(a+h, b+k)$. We shall prove that $\lambda(h, k) = D_2 D_1 f(p) hk = D_1 D_2 f(q) hk$ where p and q are two points in the rectangle with vertices $(a, b), (a+h, b), (a+h, b+k), (a, b+k)$. By symmetry it is enough to prove one of these equations:
Let $\phi(s) = f(s, b+k) - f(s, b)$. Then $\lambda(h, k) = \phi(a+h) - \phi(a) = \phi'(\theta)h = h(D_1 f(\theta, b+k) - D_1 f(\theta, b)) = hk D_2 D_1 f(\theta, \theta')$.

From these equations, $D_2 D_1 f(p) = D_1 D_2 f(q)$. Taking a sequence of h, k tending to $0, 0$, we see by continuity of the second partials that we are done. □

Proposition 1.1. *As a corollary, if f is C^r, then all of the mixed partials upto order r are equal.*

Proof. We induct on r. $r = 2$ is Clairaut. Assume truth for $2, 3, \ldots, r-1$. Then $D_i D_{j_1 \ldots j_k} f = (D_i D_{j_1})(D_{j_2} \ldots D_{j_k} f)$. Now $(D_{j_2} \ldots D_{j_k} f)$ is C^2 (why?) Thus by the C^2 Clairaut, $D_{j_1}(D_i D_{j_2} \ldots f)$ which means that i can be permuted to whichever position we want. We are done (why?) □

Here is a useful and intuitive result.

Proposition 1.2. *Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open sets. Let $f : U \to V$ and $g : V \to \mathbb{R}^p$ be C^r functions. Then $g \circ f$ is C^r for $0 \leq r \leq \infty$.*

Proof. For $r = 0$ it is a standard property. Let $r = 1$. f, g are C^1 and hence differentiable. By the chain rule, $D(g \circ f)_x = Dg_{f(x)} Df_x$. By the properties of continuity, $D(g \circ f)$ is continuous and hence $g \circ f$ is C^1.

Assume inductively that the theorem has been proven for $1, 2, \ldots, r-1$. Now $D(g \circ f)_x = Dg_{f(x)} Df_x$ is C^{r-1} by the induction hypothesis (why?). Hence, $g \circ f$ is C^r.

If $r = \infty$, then applying the result for each finite r, we see that so $g \circ f$ is C^∞. $\square$

Recall that one of the points of the one-variable chain rule was to calculate the derivatives of inverses if there existed. Indeed, assuming that $\sin^{-1}(x)$ is differentiable, since $\sin(\sin^{-1}(x)) = x$, by the chain rule, $\cos(\sin^{-1}(x))(\sin^{-1}(x))' = 1$. Thus, $(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$. Likewise,

Proposition 1.3. *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be differentiable at an interior point $a \in U$. Suppose $f(a)$ is an interior point of $f(U)$ and $f : V \to f(V)$ (where V is a neighbourhood of a) is $1-1$, onto, $f(V)$ is open, and $f^{-1} : f(V) \to V$ is differentiable at $f(a)$. Then $Df(a)$ is invertible and $Df^{-1}(f(a)) = (Df(a))^{-1}$.*

Proof. Since $f(f^{-1}(x)) = x$, by the chain rule at $f(a)$, $Df|_a Df^{-1}|_{f(a)} = I$. Hence we are done. $\square$

This is great but how did we conclude that $\sin^{-1}(x)$ was differentiable in the first place? That was a non-trivial result. More generally, if $f'(a) \neq 0$, could we have concluded that not only was f invertible but also f^{-1} was differentiable? As such, this is a silly question. Of course even $\sin(x) : \mathbb{R} \to [-1, 1]$ is not invertible! However, if we restrict ourselves to a small region like $(-\pi/2, \pi/2)$, then yes it is invertible. So perhaps we should ask whether f (assumed to be C^1 from $\mathbb{R}$ to $\mathbb{R}$) is *locally* invertible near a if $f'(a) \neq 0$ and whether the *local* inverse f^{-1} is differentiable. Indeed, if $f'(a) \neq 0$, f is monotonic near a and hence locally invertible (and the image of f of a small neighbourhood is open by the intermediate value theorem).