

MA 200 - Lecture 14

1 Recap

1. Stated and proved a generalisation of Lagrange's multipliers.
2. Injective derivative theorem.
3. A 1 – 1 regularly parametrised curve that is not homeomorphic to its image.

2 Manifolds in \mathbb{R}^n

Now we want to make a definition at this point (which will be the right kind of level sets to consider for things like Lagrange's multipliers).

Definition of an d -dimensional C^r -manifold-without-boundary or simply, d -dimensional C^r -manifold¹: It is a subset $M \subset \mathbb{R}^n$ such that for every $p \in M$, there exists an open subset V of M , an open subset $U \subset \mathbb{R}^d$ and a C^r map $\alpha : U \rightarrow \mathbb{R}^n$ such that $\alpha(U) = V$, α is a homeomorphism, and $D\alpha_x$ has rank d for each $x \in U$. The pair (α, U) is called a *coordinate parametrisation* (or *coordinate chart* or a *coordinate patch*. However, it is more common to call (α^{-1}, V) as a coordinate chart, but whatever).

Note that the same subset can be studied using different parametrisations (and that is the key point of defining manifolds: that the choice of coordinate parametrisations is upto our convenience/caprice).

Examples and non-examples:

1. The graph of $y = |x|$ is not an example of a manifold in \mathbb{R}^2 because if it were, then near the origin, there is a coordinate parametrisation $\alpha(t) = (x(t), y(t))$ such that $y(t) = |x(t)|$ and $\alpha(0) = (0, 0)$. Thus $y^2 = x^2$ and hence $yy' = xx'$. Since it is a coordinate parametrisation, either $x' \neq 0$ and therefore, y' is not continuous at $t = 0$ (why?) or $y' \neq 0$ and hence x' is not continuous at $t = 0$.
2. The letter T (as a subset of \mathbb{R}^2) is not a manifold. Indeed, if it were, then it would have been locally homeomorphic to an open ball in \mathbb{R}^n even near the point of intersection of three lines. If you throw out that point, you get three connected components on one side and at most 2 on the other side, a contradiction.

¹Caution: when you grow big and strong, you will call this beast an embedded submanifold of \mathbb{R}^n without boundary. Fortunately, just as every group is secretly a subgroup of S_n , every abstractly defined manifold-without-boundary is secretly an embedded submanifold-without-boundary of some \mathbb{R}^N

3. The graph of a C^r function over an open set is a C^r manifold.
4. The circle is an example of a 1-dimensional smooth manifold (why?). Note that the circle cannot be covered by a *single* coordinate parametrisation (because the image of a coordinate parametrisation is not compact whereas a circle is).
5. Any regular level set is an example of an $n - 1$ -dimensional manifold (why?).

In other words, if f attains an extremum on a manifold-without-boundary, and if α is a coordinate parametrisation, then $f \circ \alpha$ attains an *unconstrained* local extremum and hence its gradient is zero. This is the real point of Lagrange's multipliers.

We will return to manifolds much later (because they are the right objects for generalising the fundamental theorem of calculus to higher dimensions).

3 Taylor's theorem and the second derivative test

So far, we have only seen how to calculate global extrema but have no means of recognising whether a local extremum is a local maximum or a local minimum. In one-variable calculus, we have the famous second-derivative test (the following statement can be strengthened significantly, but we don't need to do so for our purposes):

Theorem 1. *Let $I \subset \mathbb{R}$ be open, and $f : I \rightarrow \mathbb{R}$ be a function that is C^2 in a neighbourhood of a . If $f'(a) = 0$ and $f''(a) < 0$, a is a local maximum (and likewise for local minima). Conversely, if a is a local maximum, then $f'(a) = 0$ and $f''(a) \leq 0$.*

We have already proven that if a is a local extremum, $f'(a) = 0$ (even in higher dimensions). To get further information, we need to approximate f better (than the linear approximation). To this end, we have Taylor's theorem in one-variable:

Theorem 2. *Let $U \subset \mathbb{R}$ be an open set and $f : U \rightarrow \mathbb{R}$ a C^k function on U . Let $a \in U$ and $|h| < \epsilon$ such that $(a - \epsilon, a + \epsilon) \in U$. Then the polynomial $p_{k,a}(h) = f(a) + f'(a)h + \frac{f''(a)h^2}{2!} + \dots + \frac{f^{(k)}(a)h^k}{k!}$ is the unique polynomial of degree $\leq k$ such that $\lim_{h \rightarrow 0} \frac{f(a+h) - p_{k,a}(h)}{h^k} = 0$. Moreover, if f is C^{k+1} , then $f(a+h) = p_{k,a}(h) + \frac{f^{(k+1)}(\eta)h^{k+1}}{(k+1)!}$, where η lies between a and $a+h$.*

Proof. We prove uniqueness first. Suppose p_1, p_2 are two such polynomials of degree $\leq k$. Then $\frac{p_1 - p_2}{h^d}$ goes to 0 as $h \rightarrow 0$ for all $d \leq k$. Assume that the first non-zero coefficient of $p_1 - p_2$ is that of h^d . Then we get a contradiction.

Now let $g(h) = f(a+h) - p_{k,a}(h)$. Note that g is C^k on a neighbourhood of 0, and $g^{(i)}(0) = 0$ for all $0 \leq i \leq k$ (why?) For $k = 1$, we are done easily by definition of the derivative. Assume Taylor's theorem for $1, 2, \dots, k-1$. We apply this induction hypothesis to $g'(h)$. Hence, $\frac{g'(h)}{h^{k-1}} \rightarrow 0$. Now $g(h) = g'(\zeta_h)h$ (by LMVT) and hence $\frac{g(h)}{h^k} = \frac{g'(\zeta_h)}{\zeta_h^{k-1}} \frac{\zeta_h^{k-1}}{h^{k-1}}$ which goes to 0 by the squeeze rule.

Now we prove the remainder formula. For $k = 0$ it is easy (by LMVT). Hence, assume the truth of this statement for $0, 1, 2, \dots, k-1$. For k , apply the induction

hypothesis to $g'(h)$ to conclude that $g'(t) = \frac{g^{(k+1)}(\zeta_t)}{k!} t^k$. Considering $g(t)$ and t^{k+1} and using Cauchy's Mean Value Theorem, we see that $g'(c)h^{k+1} = (k+1)c^k g(h)$. Hence $g(h) = \frac{g^{(k+1)}(\theta)h^{k+1}}{(k+1)!} = \frac{f^{(k+1)}(\theta_h)h^{k+1}}{(k+1)!}$. \square

Actually, Taylor's theorem holds even without the assumption of being C^k . In fact, k -times differentiable is good enough. But to prove such a thing, we need to use L'Hopital's rule.

Here is the proof of the second-derivative test: By Taylor, $f(a+h) = f(a) + f'(a)h + \frac{f''(\zeta)h^2}{2!}$ which means that $f(a+h) - f(a) = \frac{f''(\zeta_h)h^2}{2!}$. Since f is C^2 , if $f''(a) > 0$, then in neighbourhood of a , $f''(x) > 0$. Thus, $f(a+h) - f(a) > 0$ in a neighbourhood of $h = 0$ and hence we are done. \square