# MA 200 - Lecture 2

## 1 Recap

Discussed why to care about multivariable calculus. Reviewed linear algebra (vector spaces, dimension, basis, linear maps, matrices, rank, determinant, inner products, and norms).

## 2 Review of (real) linear algebra

Here are two (of course equivalent) norms on the space of matrices:

1. The operator norm: $\|A\|_{op} := \sup_{\|x\|_{l^2}=1} \|Ax\|_{l^2}$. (Why is this a norm?) Note that if $A$ is diagonalisable with *orthonormal* eigenvectors $e_i$ forming a basis, then $\|Ax\|_{l^2} = \|\sum_i \lambda_i x_i e_i\| = \sqrt{\sum_i |\lambda_i|^2 x_i^2} \leq \max_i |\lambda_i|$. In this case, the operator norm is $\max_i |\lambda_i|$. In general, it may not be the case even if $A$ is diagonalisable! For instance, take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

2. The Frobenius/Hilbert-Schmidt norm: $\|A\|_{HS}^2 := \sum_{i,j} ||a_{ij}|^2 = tr(A^T A)$. This norm is the usual inner product norm pretending that the space of matrices is $\mathbb{R}^{mn}$.

Both of these matrix norms satisfy $\|AB\| \leq \|A\|\|B\|$. As a consequence, if $A$ is invertible, $1 \leq \|A\|\|A^{-1}\|$.

## 3 Review of topology of $\mathbb{R}^n$

The inner product norm induces a metric in the sense of metric spaces, i.e., $d(x, y) = d(y, x), d(x, z) \leq d(x, y) + d(y, z), d(x, y) \geq 0$ with equality iff $x = y$. Recall that once we have a way to talk about distances, we can define open balls $B_a(r)$. Once we have these, we can talk of open sets, interiors, and closed sets (including limits points and closure). We can also talk of convergence of sequences: $d(x_n, x) < \epsilon$ whenever $n > N$. Moreover, every closed and bounded set is compact (and vice-versa), i.e., every open cover has a finite subcover, and equivalently, every sequence has a convergent subsequence. Hopefully you did connected sets too: A set is connected iff it cannot be written as a disjoint union of two relatively open subsets. Moreover, a set in $\mathbb{R}$ is connected iff it is an interval.

Consider a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Then $\lim_{x\to x_0, x\in U} f(x) = L$ if $\|f(x) - L\| < \epsilon$ whenever $0 < \|x - x_0\| < \delta$ and $x \in U$. One can prove that this limit can be equivalently defined using sequences. A function is said to be continuous at $x_0$ if $\lim_{x\to x_0, x\in U} f(x) = f(x_0)$. If we restrict the domain of a continuous function, it still remains continuous (and likewise for limits). (But beware! if you enlarge the domain, the function might stop being continuous!) Equivalently, a function is continuous iff the inverse image of an open set is open.

Continuous functions take compact sets to compact sets. As a consequence, we have the extreme-value theorem. Also, uniform continuity. Continuous functions also take connected sets to connected sets. Continuous functions satisfy various properties (sum, product, quotient (these three hold for limits too), and composition). One can come up several examples (using continuity/limit laws. Polynomials for instance are continuous) and non-examples (using sequences along different paths). Another example of a continuous function:

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

# 4 Derivatives

Recall that in one-variable calculus, $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$. In more than one variable, unfortunately, this naive definition cannot work (because we cannot divide by a vector). A reasonable substitute is the notion of a directional derivative of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ at an *interior* (why?) point $a \in U$ along a vector $\vec{v}$: $\nabla_v f(a) = \frac{df(a+tv)}{dt}|_{t=0}$. (Caution: When $v = 0$, the name "directional derivative" is somewhat of a misnomer. Moreover, since $\nabla_{cv} f(a) = c\nabla_v f(a)$, again this name is not completely appropriate.) Examples:

1. When $v = e_i$, the resulting directional derivative is called the partial derivative of $f$ w.r.t $x_i$ and is denoted as $\frac{\partial f}{\partial x_i}$. This quantity can be calculated easily using the various rules for one-variable differentiation. (Tidbit: The laws of nature are partial differential equations, i.e., equations involving partial derivatives.)

2. One can have directional derivatives at all points in all directions: Polynomials for instance (note that this is a one-variable question!)

3. It is certainly possible to have directional derivatives along some directions and not along some others:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

   has directional derivatives at $(0,0)$ along $e_1$ for instance but not along $e_1 + e_2$.

4. It is possible to have directional derivatives along all directions at all points in a domain and yet fail to be even continuous!

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The last example illustrates that the notion of a directional derivative is not a good enough notion. Indeed, differentiability is a "nicer" condition than continuity. It must imply continuity at the very least! (Another problem (albeit less important) with directional derivatives is that, apparently, we need to keep track of *infinitely* many numbers (one for each direction) at even a *single* point of the domain to understand how quickly the function changes at that point.)