MA 200 - Lecture 16

1 Recap

1. Defined the Riemann integral and stated the integrability criterion $U(P, f) - L(P, f) < \epsilon$.

2 Measure zero sets and integrability (from Munkres)

We proved that piecewise-constant functions are integrable. So what went wrong with the Dirichlet function? It isn't just that there were infinitely many discontinuities (you will see an example in your HW). It is that probabilistically speaking, if you throw a dart at them, you are guaranteed to hit a discontinuity (since probability is presumably vaguely related to length, we already see a problem). Lebesgue proved a criterion that decided Riemann integrability (based on this probabilistic intuition).
Instead of probabilities, let us try to define when a set has zero volume:
Def: Let $A \subset \mathbb{R}^n$. It is said to have measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$ there is a cover of A by countably many closed rectangles $Q_1, \ldots,$ such that $\sum_i v(Q_i) < \epsilon$. (This is abbreviated as "the total volume of this cover is less than ϵ".)
What if A is a rectangle itself? At least if we cover a closed rectangle Q by finitely many rectangles $Q_1, \ldots, Q_k$, then is $v(Q) \leq \sum_{i=1}^{k} v(Q_i)$? Thankfully yes: Choose a large rectangle Q' containing $Q_1, \ldots, Q_k$. The end points of $Q, Q_1, \ldots, Q_k$ (and those of Q') form a partition of Q'. In particular, the intersection of this partition with Q or Q_1 or $\ldots$ is a partition of each of them. Thus each of these rectangles is a union of some subrectangles from the bigger partition. We conclude (using the previous results) that $v(Q) = \sum_{R \subset Q} v(R)$. Now each such R is in at least one of the Q_i (because the Q_i form a cover). Thus, $\sum_{R \subset Q} v(R) \leq \sum_{i=1}^{k} \sum_{R \subset Q_i} v(R)$ (finite summations can be done in any order by induction). Now again, $\sum_{R \subset Q_i} v(R) = v(Q_i)$. Hence we are done.
Here is an example of a measure zero set: The rationals in $[0, 1]$ have measure zero: Indeed, they are countable. So enumerate them as $a_1, a_2, \ldots$. Now consider the cover $[a_i - \frac{\epsilon}{2^i}, a_i + \frac{\epsilon}{2^i}]$. The total volume is less than ϵ. In fact, this argument works for any countable subset of $\mathbb{R}$. Even more generally, suppose A is a countable subset of $\mathbb{R}^n$. Then enumerate it as before and consider the cover $[a_{i1} - \frac{\epsilon}{2^i}, a_{i1} + \frac{\epsilon}{2^i}] \times [a_{i2} - \frac{\epsilon}{2^i}, a_{i2} + \frac{\epsilon}{2^i}] \ldots$.
The above example indicates that we genuinely need countably many sets and cannot do with finitely many. Indeed, rationals in $[0, 1]$ have measure zero. If we could cover them by finitely many rectangles whose total volume is $< \frac{1}{2}$, then by density of rationals, $[0, 1]$ is covered by these rectangles but its volume is 1.
Here are a few properties:

1. If $B \subset A$ and A has measure zero, then so does B (almost by definition).

2. Let $A = \cup_i A_i$ where A_i have measure zero. Then so does A: Indeed, cover A_i by rectangles R_{ij} whose total volume is less than $\frac{\epsilon}{2^i}$. Then the R_{ij} cover all of A. Now $\sum_i \sum_j v(R_{ij}) < \sum_i \frac{\epsilon}{2^i} < \epsilon$. Enumerate the rectangles R_{ij} so that $\sum_n v(Q_n) = \sum_i \sum_j v(R_{ij})$ (but it actually does not matter what order you use because this series is absolutely summable and Fubini's theorem holds).

3. The closed rectangles in the definition of measure zero can be replaced by their interiors (i.e., open ones): Of course, if A is covered by open rectangles whose total volume is less than ϵ, then their closures also cover A and hence we are done. Conversely, if A is covered by closed rectangles whose total volume is less than $\frac{\epsilon}{10}$, then enlarge the i^{th} closed rectangle by scaling it by a factor of 2. The total volume is still less than ϵ.

4. If Q is a rectangle (not a point) in $\mathbb{R}^n$, then the boundary has measure 0 but Q does not: For each boundary face, enlarge it by $\frac{\epsilon}{4^n}$. The sum of the volumes of these enlarged rectangles is less than ϵ and covers the boundary. (These are finitely many rectangles. To get a countable collection, simply choose the other rectangles to be points.)
   As for Q not having measure 0, if it did, cover it with finitely many open rectangles (by compactness and the previous step) whose total volume is less than $v(Q)$. This is a contradiction.

Now we have enough machinery to prove Lebesgue's theorem:

Theorem 1. *Let $Q \subset \mathbb{R}^n$ be a closed rectangle and $f : Q \to \mathbb{R}$ be a bounded function. Let $D \subset Q$ be the set of discontinuities of f. Then f is R.I iff D has measure zero in $\mathbb{R}^n$.*

Proof. Let $|f(x)| \leq M \ \forall \ x \in Q$.

1. If D has measure 0: Roughly speaking, we shall cover D with countably many rectangles of small total volume, and we shall cover the other points by rectangles where $M_R - m_R$ is small. Since Q is compact, only finitely many of all of these rectangles are necessary and using the endpoints of these finitely many rectangles, we shall produce a partition P such that $U(P, f) - L(P, f) < \epsilon$.
   Cover D by open rectangles $Int(Q_1), \ldots$ of total volume less than ϵ' (which we shall see later ought to be chosen to be $\frac{\epsilon}{2M+2v(Q)}$). If a is a point where f is continuous, choose an open rectangle $Int(Q_a)$ containing a such that $|f(x) - f(a)| < \epsilon'$ when $x \in Q_a \cap Q$ (the closed rectangle). Then $Int(Q_i), Int(Q_a)$ cover Q. Since Q is compact, a finite subcollection (that we shall relabel if necessary) $Int(Q_1), \ldots, Int(Q_k), Int(Q_{a_1}), \ldots Int(Q_{a_l})$ cover Q (these need not cover all the discontinuities or all continuities). Replace Q_{a_j} and Q_i by their intersections with Q. Take a partition P given by the endpoints of each component interval of Q_i and Q_{a_j}. Then each closed rectangle is a union of sub-rectangles of this partition. Now every sub-rectangle R is either in Q_i or in Q_{a_j}. If it is in the former, then $(M_R - m_R)v(R) \leq 2Mv(R)$ and if it is in the latter, $(M_R - m_R)v(R) < 2\epsilon' v(R)$. Thus, $U(P, f) - L(P, f) \leq 2M\epsilon' + 2\epsilon' v(Q) = (2M + 2v(Q))\epsilon' = \epsilon$ if $\epsilon' = \frac{\epsilon}{2M+2v(Q)}$.

2. If f is R.I: To be continued.

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