MA 200 - Lecture 11

1 Recap

1. Finished the proof of IFT.
2. Stated and proved IFT for 2 variables.

2 Implicit function theorem

More generally, we have

Theorem 1 (Implicit function theorem)**.** *Let $f(x,y) : \tilde{U} \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be a C^r function (and $\tilde{U}$ be an open set). Suppose $f(a,b) = c$ and $D_y f_{a,b}$ is invertible. Then there exists a connected neighbourhood A of a and a connected neighbourhood B of b such that $A \times B \subset \tilde{U}$, and a* unique *$C^r$ function $g : A \to \mathbb{R}^n$ such that $f(x,y) = c$ and $(x,y) \in A \times B$ if and only if $y = g(x)$.*

Proof. As before, consider $H(x,y) = (x, f(x,y))$. This function is also C^r (why?). Moreover, $DH_{(a,b)} = \begin{pmatrix} I & 0 \\ D_x f_{a,b} & D_y f_{a,b} \end{pmatrix}$. This is invertible (why?) Now $H(a,b) = (a,c)$. The IFT shows that there are (connected) neighbourhoods $U \times V \subset \tilde{U}$ and W of (a,b) and (a,c) respectively such that $H : U \times V \to W$ is a C^r diffeomorphism. This means that $H^{-1}(p,q) : W \to U \times V$ exists and is a C^r map. Thus $x = h_1(p,q)$ and $y = h_2(p,q)$ such that $H(h_1, h_2) = (p,q)$, i.e., $h_1(p,q) = p$ and $f(h_1(p,q), h_2(p,q)) = q$. So consider small enough neighbourhoods (which are connected) $A \subset U$ of a, B of b, C of c such that $A \times C \subset W$, and $B \subset h_2(A \times C)$. Now $y = h_2(x,c)$ does the job. (In fact, it also shows that y is C^r function of x, c taken together.)
As for uniqueness, suppose $g_0 : A \to \mathbb{R}^n$ is another such function. Then $g_0(a) = g(a)$. Moreover, $H(x, g_0(x)) = (x, f(x,y))$. Since H^{-1} (which is locally defined) is unique, g_0 coincides with g in a neighbourhood of a (why?) The set of points in A where they coincide is closed (why?) It is non-empty and by this argument, it is open as well. Hence, by connectedness it is all of A. □

Remark: There is nothing special about the last few coordinates. You are allowed to permute them, i.e., solve for some in terms of the others.
Examples/Non-examples:

1. Let $f(x, y) = x^2 - y^3$. Then $\nabla f(0, 0) = (0, 0)$. Therefore, it appears that the origin is a problematic point. (The graph looks rather singular.) Indeed, there is no C^1 way to solve for x in terms of y or vice-versa (we can solve for y in terms of x uniquely but the expression is not C^1!)

2. $f(x, y) = y^2 - x^4$. Here, we cannot solve for y in terms of x uniquely, but the two solutions are smooth. Elsewhere, there is no such problem.

3. If $A \in Mat_{2\times2}(\mathbb{R})$, when is there B such that $B^2 = A$? The answer is NO in general even if the eigenvalues are non-negative (standard non-diagonalisable matrix). On the other hand, there exists a neighbourhood of the identity such that every matrix in this neighbourhood has a square root: $F(A) = A^2$ is C^1 and $DF_I(H) = 2H$, which is invertible and hence by IFT, we are done.

4. Consider a C^1 function $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Suppose $\nabla f(a) \neq \vec{0}$ and $f(a) = 0$. Then $\frac{\partial f}{\partial x_1}(a) \neq 0$ WLOG. Therefore, by the implicit function theorem, near a, we can solve $f(x) = 0$ for $x_1 = g(x_2, \ldots, x_n)$ in a C^1 manner (what does this mean precisely?) Suppose $\gamma(t) : I \subset \mathbb{R} \to U$ is a C^1 map such that $f(\gamma(t)) = 0 \, \forall \, t$. Then $\langle \nabla f(a), \gamma'(0) \rangle = 0$. Moreover, given *any* vector v such that $\langle \nabla f(a), v \rangle = 0$, we see that $\gamma(t) = (g(a_2 + v_2 t, a_3 + v_3 t, \ldots), a_2 + v_2 t, \ldots)$ is C^1, lies on the level set, and $\gamma'(0) = v$ (why?). Hence, $\nabla f(a)$ is in a reasonable sense, a "normal" to the level set. The tangent plane at a is $\langle \nabla f(a), \vec{r} - \vec{a} \rangle = 0$. This definition coincides with the definition given earlier for a graph when $f(x) = x_1 - g(x_2, \ldots, x_n)$ (why?).