HW 12

1. Let $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ be linearly independent vectors. Let V be the $n \times (n-1)$ matrix $V = [v_1\ v_2\ \ldots\ v_{n-1}]$. Let $c = \sum_i c_i e_i$ where $c_i = (-1)^{i-1} \det(v_1\ v_2 \ldots\ v_{i-1}\ v_{i+1} \ldots)$. Prove that c is the unique vector satisfying the following properties.

   (a) $c \neq 0$ and $c \perp v_i\ \forall\ i$.

   (b) $\det(c\ v_1\ v_2 \ldots) > 0$.

   (c) $\|c\|^2 = \det(X^T X)$.

2. Complete the proof of the fact that a hypersurface is orientable iff it has a C^r unit normal vector field.

3. Suppose $M \subset \mathbb{R}^n$ is a C^r k-dimensional manifold-with-nonempty-boundary. Let $\vec{n} \in \mathbb{R}^n$ be a vector. Define $\vec{n}$ to be outward-pointing at a boundary point $p \in \partial M$ if for there exists a boundary parametrisation $\alpha : U \subset \mathbb{H}^k \to \mathbb{R}^n$ such that $\vec{n}(p) = -\frac{\partial \alpha}{\partial x_k}(\alpha^{-1}(p))$.

   (a) Choose an outward pointing vector n at a point $p \in \partial M$. Define a boundary parametrisation β to be compatible with the outward pointing direction if $\det(n\ \frac{\partial \beta}{\partial x_1}(p)\ \ldots) > 0$. Prove that this notion is independent of n.

   (b) Suppose M is oriented. Prove that the set of restrictions of all boundary parametrisations compatible with the outward pointing direction form an orientation for ∂M.

   (c) Prove that this orientation is compatible with the standard orientation defined in the class.

4. Find an explicit orientation-compatible collection of parametrisations for the closed unit ball in $\mathbb{R}^4$ and an explicit collection of parametrisations for its boundary that are orientation-compatible with the standard orientation.