HW 7

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^r (where $r \geq 1$) function. Suppose $f^{-1}(0) \neq \phi$ and whenever $x \in f^{-1}(0)$, $\nabla f(x) \neq 0$. Then prove that $f^{-1}(0)$ is a C^r-manifold with dimension $n-1$.

2. In the Taylor theorem in multivariable calculus, prove that the Taylor polynomial is the unique such polynomial upto degree k.

3. Find the local extrema of $f(x, y) = 2x^2 - 2x^2y^2 + 2y^2 + 24y - x^4 - y^4$ on $\mathbb{R}^2$ and classify them.

4. Write the Taylor polynomial around $(0, 0)$ of degree 3 (with justification) of $f(x, y) = 5x^4y + 3xy - \pi xy^3 + \pi^2x^2y^2 + ex^2y + 100xy^2 - \frac{1}{100}xy + \frac{e}{\pi^e}x + 42$.

5. This is a vague question: If the Hessian of a smooth function is not invertible at a, and the third derivatives are all zero at a point where $\nabla f(a) = 0$, how can you hope to classify the point as a local extremum or a saddle point (under suitable hypotheses)?