NOTES FOR 12 SEPT (THURSDAY)

1. Recap

(1) Proved that between f.d. spaces of the same dimension, T is non-singular iff it is onto.
(2) Expressed linear maps as matrices using bases and looked at the change of basis formula. Defined similarity of matrices.

2. Linear functionals

A linear functional on V is a linear map $F : V \to \mathbb{F}$. Here are some fundamental examples.

(1) If $V = \mathbb{F}^n$, then $F_a(\vec{v}) = \sum_i a_i v_i$ where $\vec{a} \in V$ is a linear functional (as can be seen easily). Note that in the case where $\mathbb{F} = \mathbb{R}$, the functional is simply the dot product. Here is a fun fact : If $F_a(\vec{v}) = 0 \ \forall \ \vec{a}$, then $\vec{v} = \vec{0}$. Indeed, choose $a = e_i$. Note that the notion of a dot product (which we will discuss later) does not make sense for vector spaces over all fields. Nonetheless, this linear functional is a useful substitute. Another interesting point : $dim(Ker(F_a)) = n - 1$ whenever $a \neq 0$. (The kernel is called a hyperspace/hyperplane.) Indeed, the range is all of $\mathbb{F}$ (why ?) and nullity-rank tells us what we want. (Actually ,this statement does not need the nullity-rank theorem.)
(2) The Trace map $tr : Mat_{n\times n}(\mathbb{F}) \to \mathbb{F}$ given by $tr(A) = \sum_i A_{ii}$ is a linear functional. Note the following interesting fact : $tr(AB) = \sum_{i,j} A_{ij}B_{ji} = \sum_{i,j} B_{ji}A_{ij} = tr(BA)$. (As a further consequence, there are no two matrices X, P such that $[X, P] = I$.)
(3) Let V be the space of all functions $f : S \to \mathbb{F}$ where S is any set. Then $E_t : V \to \mathbb{F}$ given by $E_t(f) = f(t)$ is a linear functional (called the evaluation map).
(4) Let $V[a, b]$ be the space of all continuous functions $f : [a, b] \to \mathbb{R}$. Then $F(f) = \int_a^b f(x)dx$ is a linear functional.

The collection of all linear functionals $L(V, \mathbb{F})$ on V is denoted as V^*. It is a vector space called the dual space of $\mathbb{F}$ of dimension $dim(V)$ (if V is finite-dimensional). Explicitly, the functionals $w_i(e_j) = \delta_{ij}$ form a basis (known as the basis dual to e_i). For every vector $v \in V$, $v = \sum_i w_i(v)e_i$. Moreover, if $w \in V^*$, $w = \sum_i w(ve_i)w_i$. Here is an interesting example : Let V be the space of real polynomial functions of degree ≤ 2. Let t_1, t_2, t_3 be distinct real numbers. Consider the evaluation functions ev_{t_i}. They are are linearly independent. Indeed, applying the functional equation $\sum_i c_i ev_{t_i} = 0$ to the vectors $1, t, t^2$ we get

$$\begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \\ t_1^2 & t_2^2 & t_3^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.1}$$

By Gaussian elimination it is easy to see that $c_i = 0 \ \forall \ i$. Hence ev_{t_i} form a basis. Is there are a basis p_i of V such that ev_{t_i} is its dual basis ? That is, $ev_{t_i}(p_j(t)) = p_j(t_i) = \delta_{ij}$? Note that $p_1(t)$ has two roots t_2, t_3 and hence we can try $p_1(t) = A(x - t_2)(x - t_3)$ where $A = \frac{1}{(t_1-t_2)(t_1-t_3)}$ works. Likewise for the other p_i. Interestingly enough, $p(t) = \sum_i p(t_i)p_i(t)$. Therefore, there exists exactly one quadratic which satisfies $p(t_i) = c_i$ given by the above formula.

Since linear functionals seem to be useful substitutes for "dot products" (whatever they may be) here is a definition : If V is a vector space and $S \subset V$ is a subset, the annihilator of S is the set $S^0 \subset V^*$ consisting of f such that $f(v) = 0 \ \forall \ v \in S$. Note that S^0 is always a subspace (regardless of the status of S). The smaller S is, the larger S^0 is.

Theorem 2.1. *Let V be a finite-dimensional vector space and $W \subset V$ be a subspace. Then $dim(W)+dim(W^0) = dim(V)$.*

Proof. Let $e_1, \ldots, e_w$ be a basis of W extended to $e_1, \ldots, e_n$ - a basis of V. Then consider $e^*_{w+1}, \ldots, e^*_n$. Clearly the subspace $S \subset V^*$ spanned by them annihilates W. Moreover, since for every $f \in V^*$, $f = \sum_i f(e_i)e^*_i$, we see that if $f(W) = 0$, then f is in S. Hence, $S = W^0$. □

The proof of this theorem shows that if W is a k-dimensional subspace of V, then W is the intersection of $n - k$ hyperplanes. Indeed, the hyperplanes given by $e^*_{w+i}(v) = 0$ intersect precisely in W. Another corollary is : If $W_1, W_2 \subset V$ are subspaces of a finite-dimensional vector space V, then $W_1 = W_2$ iff their annihilators are equal. Indeed, if $W_1 = W_2$, clearly their annihilators are equal. Conversely, suppose there exists a vector $v \in W_1 \cap W_2^c$. Then, extend v to a basis of W_2 and V. The linear functional v^* kills W_2. However, $v^*(v) = 1$. A contradiction (because the annihilators are assumed to be the same).

We can now look at linear equations from the perspective of linear functionals. Indeed, if $\sum_j A_{ij}x_j = 0$, let f_i be the functional defined as $f_i(x) = \sum_j A_{ij}x_j$. Then the solution space of the linear equations is simply the subspace annihilated by all the f_i. Moreover, $f_i = \sum_j A_{ij}e^*_j$. So the row space is a subspace of V^* spanned by f_i. Looking at it from a dual point of view, given vectors $\alpha_i = (A_{i1}, \ldots, A_{in})$, the condition that a linear functional $f(x) = \sum_i c_i x_i$ is annihilates α_i is indeed $\sum_i A_{ij}c_j = 0$. Thus, row-reduction gives us an algorithm to find the annihilator of the subspace spanned by a set of vectors in $\mathbb{F}^n$.