NOTES FOR 13 AUG (TUESDAY)

1. Recap

(1) Defined invertible matrices and proved that the product of an arbitrary number of invertible matrices is invertible. (As a consequence, a product of elementary row matrices is invertible.) Also, proved that if A is invertible, its left and right inverses are unique and equal.
(2) One can that A is left invertible iff it is row equivalent to the identity iff A is a product of elementary row matrices. Using this one can prove that A is invertible iff $AX = 0$ has the trivial solution iff $AX = Y$ has a unique solution for every Y.
(3) As a consequence, $A = A_1 \ldots A_k$ is invertible iff each of the A_i is so.
(4) Defined vector spaces and proved some properties of them.

2. Vector spaces

Def : A vector β is said to be a linear combination of $v_1, \ldots, v_n$ with coefficients α_i if $\beta = \sum_i \alpha_i v_i$. Here are some examples and non-examples.

(1) If $\mathbb{F}$ is a field, then $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ by component-wise addition and scalar multiplication of each coordinate by field elements. For instance $\mathbb{R}^n$ is the usual vector space consisting of "vectors" where vector addition has the usual interpretation.
(2) Note that the same set can be a vector space over different fields. For instance, $\mathbb{C}^n$ is a vector space over $\mathbb{R}$. Likewise, $\mathbb{R}$ is a vector space over $\mathbb{Q}$.
(3) The set of $m \times n$ matrices with elements from the field form a vector space under $(A + B)_{ij} = A_{ij} + B_{ij}$ and $(cA)_{ij} = cA_{ij}$. The additive inverse is $-A_{ij}$ and the zero element is $0_{ij} = 0$.
(4) Let S be any set and V be the set of functions $f : S \to \mathbb{F}$. Define $(f + g)(s) = f(s) + g(s)$ and $(cf)(s) = cf(s)$. V is a vector space over $\mathbb{F}$.
(5) Define the set of degree-d polynomials with coefficients in $\mathbb{F}$ as the subset $(a_0, a_1, \ldots, a_d, 0, 0 \ldots) \in \mathbb{F}^\infty$ where $a_d \neq 0$. Addition is component wise. We define $x = (0, 1, 0, \ldots)$ and multiplication of polynomials as $(a.b)_i = \sum_{j+k]i} a_j b_k$. So $a(x) = \sum_i a_i x^i$. The set of degree-d polynomials is NOT a vector space. However, the set of degree at most-d polynomials is a vector space.
(6) A polynomial of degree-d defines a function $f : \mathbb{F} \to \mathbb{F}$ as $f(s) = \sum_i a_i s^i$. The set of polynomial functions of degree at most d also forms a vector space. Interestingly enough, polynomials are not necessarily determined by polynomial functions for finite fields. For instance, $x + x^2$ is 0 as a function over $\mathbb{Z}_2$. More generally, $x + x^2 + x^3 + \ldots + x^n = 0$ over a field of size n.
(7) The set of all continuous $f : \mathbb{R} \to \mathbb{R}$ is a vector space. Likewise for differentiable functions.
(8) The set of positive functions on $\mathbb{R}$ is not a vector space.
(9) The set of colours in the RGB format is not a vector space.
(10) $\mathbb{Z}^n$ is not a vector space.
(11) The set of twice differentiable functions y solving $y'' + P(x)y' + Q(x)y = 0$ is a vector space whereas $y'' + P(x)y' + Q(x)y = R(x)$ is not a vector space.

A subspace $S \subset V$ of a vector space is a set such that is a vector space in its own right under the induced operations from V. In particular, it is closed under addition, additive inverses and scalar multiplication. (These conditions are also sufficient.) Actually, it is easy to see that it is enough to be closed under $cv + w$ where $v, w \in S$. Here are examples and non-examples.

(1) 0 is a subspace of any vector space.
(2) In $\mathbb{F}^n$, $x_1 + x_2 + \ldots + x_n = 0$ is a subspace whereas $\sum_i x_i = 1$ is not.
(3) The set of twice differentiable functions $y : \mathbb{R} \to \mathbb{R}$ solving $y'' + P(x)y' + Q(x)y = 0$ is a subspace of all twice differentiable functions. Likewise, the set of polynomial functions is a subspace of the set of all functions.
(4) Symmetric matrices form a subspace of $M_{n\times n}(\mathbb{F})$.
(5) Hermitian matrices form a real subspace of $M_{n\times n}(\mathbb{C})$ (Not a complex subspace though).
(6) The solutions of $AX = 0$ form a subspace. Indeed, $A(cv + w) = cAv + Aw = 0$.
(7) The intersection of any collection of subspaces of V is a subspace. Indeed, if v, w are in all the subspaces, so is $cv + w$. It is not empty because it contains 0.
(8) Let $S \subset V$. The subspace W_S spanned by (or generated by) S is defined as the intersection of all subspaces of V containing S.

Prop : The subspace W_S spanned by a non-empty subset S is the set of all linear combinations of vectors in S.

Proof. Denote the set of all linear combinations of vectors in S by L_S. Obviously $S \subset L_S \subset W_S$. We just need to show the converse by showing that L_S is a subspace (and hence $W_S \subset L_S$). Indeed, $c\sum_i \alpha_i v_i + \sum_j \beta_j w_j$ is a linear combination of vectors in S.

Def : If $S_1, \ldots, S_k \subset V$, then the set of all sums $\sum_i v_i$ where $v_i \in S_i$ is called the sum of the subsets S_i. Clearly, if the S_i are subspaces, the sum is a subspace. In fact, it is the subspace spanned by $\cup_i S_i$. One can come up with more examples of subspaces through intersections and spanning (especially of $\mathbb{F}^n$ and $M_{m\times n}$). □