# NOTES FOR 15 OCT (TUESDAY)

## 1. Recap

(1) Defined eigenvalues, eigenvectors, characteristic polynomial, geometric and algebraic multiplicities, and diagonalisability.
(2) Observed that infinite-dimensions are subtle, eigenvalues need not always exist even in finite-dimensions, but always do over complex numbers.
(3) Proved Cayley-Hamilton for diagonalisable matrices.
(4) Not every matrix is diagonalisable. However, it appears that real symmetric matrices might be diagonalisable. (Explaining our change of variables in our ODE.) Non-symmetric ones can also be diagonalisable.
(5) Can every matrix/linear map be brought to an upper triangular form at least ? (For ODE purposes, that suffices.)

## 2. Eigenvalues and Eigenvectors

We give a partial answer to which matrices are diagonalisable.

**Proposition 2.1.** *Let $A \in Mat_{n\times n}(\mathbb{F})$ have $k$ distinct eigenvalues in $\mathbb{F}$. Then the corresponding eigenvectors in $\mathbb{F}^n$ are linearly independent.*

*Proof.* Suppose $\sum_i c_i e_i = 0$. At this point there are two ways to proceed.

(1) Induction on $k$ : Now $\sum_i c_i \lambda_i e_i = 0$ and hence $\sum_i c_i(\lambda_i - \lambda_1)e_i = 0$. Hence, a linear combination of $e_i$ (where $i \neq 1$) is 0. By the induction hypothesis, this means that $c_i = 0 \ \forall 2 \leq i \leq k$. Hence, $c_1 = 0$.
(2) Determinants : Clearly $\sum_i c_i \lambda_i^l e_i = 0$ for all $l$. Solving for $c_i$ using the Vandermonde determinant we see that $c_i = 0$.

□

As a corollary, if $A$ has $n$ distinct eigenvalues in $\mathbb{F}$, it is diagonalisable over $\mathbb{F}$. As a consequence, "most" complex matrices are diagonalisable. We give another answer to the question now.

**Proposition 2.2.** *Let $T : V \to V$ have $n = dim(V)$ eigenvalues in $\mathbb{F}$. Then, $gM \leq aM$ for every eigenvalue of $A$ and equality holds for all eigenvalues simultaneously iff $T$ is diagonalisable.*

*Proof.* Suppose not, i.e., there is an eigenvalue $\lambda$ such that $gM > aM$, i.e., there exists a basis of $V$ containing $e_1, e_2, \ldots, e_{gM}$ linearly independent eigenvectors corresponding to $\lambda$. Then the matrix of $T$ in this basis is of the form $\begin{bmatrix} \lambda & 0 & \ldots & * & * \\ 0 & \lambda & \ldots & * & * \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & * & * \end{bmatrix}$. The characteristic polynomial of this matrix (by induction) has $\lambda$ as a root repeated at least $gM$ many times - a contradiction. If $aM = gM$ for all eigenvalues, then since $\sum aM_i = n$, we have a basis of eigenvectors and $T$ is diagonalisable. If $T$ is diagonalisable, clearly $aM = gM$ for all eigenvalues. □

This answer is not very satisfying because it is not easy to check. Now we prove an important result about any matrix/linear map.

**Theorem 2.3.** *Let $T : V \to V$ have $n = dim(V)$ eigenvalues (counted with multiplicity) $\lambda_1 \neq \ldots \neq \lambda_i$ in $\mathbb{F}$. Then there exists a basis of $V$ such that $e_1, \ldots, e_k$ are all eigenvectors, and $e_{k+1}, \ldots, e_n$ are not eigenvectors but the matrix $A$ is upper triangular with the diagonal elements for $e_{k+1}, \ldots$ being $\lambda_1, \lambda_2, \ldots$ in that order inductively, i.e.,*

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 & * & * & \ldots & * \\ 0 & \lambda_1 & 0 & \ldots & 0 & * & * & \ldots & * \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \lambda_2 & \ldots & * & * & \ldots & * \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 & * & \ldots & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}.$$

*Proof.* We use induction on $n$. For $n = 1$, it is trivial. Assume truth for $1, 2 \ldots, n-1$. Take all the eigenvectors $e_1, \ldots, e_k$ and extend them to a basis of $V$ to get a matrix $B$ for $T$. Now consider the lower $(n-k) \times (n-k)$ matrix. Firstly, as shown earlier, the eigenvalues of this smaller matrix form a subset of the eigenvalues of $T$. Secondly, changing the basis $\tilde{e}_{k+1}, \ldots, \tilde{e}_n$ and using induction we can get the smaller matrix in the desired form. Since this change of basis does not affect $e_1, \ldots, e_k$, that part of the matrix remains unaffected. We are done. ◻

Essentially, in the proof of the above result, we took $V/Eigenspaces$ and applied induction to it. The above result can be used to make precise the notion that most complex matrices are diagonalisable. Firstly, endow $Mat_{n\times n}(\mathbb{C})$ the metric space topology arising from $\mathbb{C}^{n^2}$. Now we prove that *diagonalisable matrices form a dense subset of all matrices* : Indeed, given a matrix $A$, we see by the theorem that $A = P\tilde{A}P^{-1}$ where $\tilde{A}$ is in the upper triangular form as above. The idea is to add a small perturbation to $A$ so that its eigenvalues become distinct and hence $A$ becomes diagonalisable. It is easy to see that there is a sequence of diagonal matrices $D_i$ such that $D_i \to 0$ in the topology above and $\tilde{A} + D_i$ has distinct entries on its diagonal (and hence distinct eigenvalues because for an upper triangular matrix, the eigenvalues are the diagonal entries). If we prove that matrix multiplication (from left and right) is continuous, then $A + PD_iP^{-1} \to A$. Hence diagonalisable matrices are dense. I leave it as an exercise to prove that matrix multiplication is continuous (indeed, every entry of the new matrix is a polynomial). So we have the following beautiful consequence :

**Theorem 2.4.** *(Cayley-Hamilton) Every complex matrix $A$ satisfies $p_A(A) = 0$*

*Proof.* $p_A(A)$ is a matrix whose entries are polynomials in the entries of $A$. Hence $p_A(A)$ is continuous on the space of matrices with the topology described above. Since $p_A(A) = 0$ for diagonalisable matrices, and diagonalisable matrices form a dense subset, $p_A(A) = 0$ for all complex matrices. ◻

In the HW you will be asked to show that the above theorem holds for matrices over arbitrary fields (and in fact over commutative rings).