NOTES FOR 17 SEPT (TUESDAY)

1. Recap

(1) Defined linear functionals and gave examples.
(2) Defined the annihilator and saw that row-reduction gives us an algorithm to find the annihilator of the subspace spanned by a set of vectors in $\mathbb{F}^n$.

2. Linear functionals

(1) Let $f_1 = x_1 + 2x_2 + 2x_3 + x_4,\ f_2 = 2x_2 + x_4,\ f_3 = -2x_1 - 4x_3 + 3x_4$ be linear functionals on $\mathbb{R}^4$. Find the subspace they annihilate. (Basically, find the null space of an appropriate matrix with the coefficients of f_i as the rows.)
(2) Let $W \subset \mathbb{R}^5$ be the span of $\alpha_1 = (2, -2, 3, 4, -1), \alpha_2 = (-1, 1, 2, 5, 2), \alpha_3 = (0, 0, -1, -2, 3), \alpha_4 = (1, -1, 2, 3, 0)$. Find the annihilator. (Basically, put these as the rows of a matrix and find the null space of that matrix.)

3. Double dual

Is every basis of V^* dual to some basis of V ? To answer this, we consider V^{**}.

Theorem 3.1. *Let V be a finite-dimensional vector space. For each $a \in V$, define $E_a(f) = f(a)$ where $f \in V^*$. The map $a \to E_a$ is a linear isomorphism of V onto V^{**}.*

Proof. Note that the kernel consists of a such that $E_a(f) = 0$ for all $f \in V^*$, i.e., $f(a) = 0\ \forall\ f$. So $ann(a) = V$ but that is a problem because $dim(\{a\}) \neq 0$ if $a \neq 0$. Hence the kernel is trivial (it is non-singular). Hence, E_a is onto. □

As a corollary, every basis of V^* is the dual of some basis of V : Indeed, if $f_1, \ldots, f_n$ is a basis of V^*, let f_i^* be the dual basis in V^{**}. Then there exist $e_i \in V$ such that $E_{e_i} = f_i^*$. We claim that $e_i^* = f_i$. Indeed, $f_i(e_j) = E_{e_j}(f_i) = f_j^*(f_i) = \delta_{ij}$.
Sometimes, one simply identifies V with V^{**} without mention (using the isomorphism above).

Proposition 3.2. *Let $S \subset V$ be a subset of a finite-dimensional vector space V. Then $(S^0)^0$ is the subspace spanned by S.*

Proof. $f \in (S^0)^0$ iff $f(g) = 0\ \forall\ g \in S^0$ iff $g(f) = 0\ \forall\ g \in S^0$. In particular, $W_S \subset (S^0)^0$. But $dim(W_S) + dim(S^0) = n = dim(S^0) + dim((S^0)^0)$. Hence $W_S = (S^0)^0$. □

At this point we can give a deeper meaning to the statement that the row rank equals the column rank. Note that if $T : V \to W$ is a linear map, then we have a natural linear map (called the transpose/adjoint of T) $T^* : W^* \to V^*$ given by $T^*(f)(v) = f(Tv)$. Indeed, $(cT_1 + T_2)^*(f)(v) = f(cT_1(v) + T_2(v)) = cT_1^*(f)(v) + T_2^*(f)(v)$.

Theorem 3.3. *Let $T : V \to W$ be a linear map between vector spaces. Then $ker(T^*)$ is the annihilator of $Range(T)$. If V and W are finite-dimensional, then*
(1) *$rank(T^*) = rank(T)$*
(2) *Range of T^* is the annihilator of the kernel of T.*

Proof. $f \in ker(T^*)$ iff $T^*(f) = 0$, i.e., $f(Tv) = 0\ \forall v \in V$ iff $f(Range(T)) = 0$ iff f is in the annihilator of $Range(T)$. Now assume that V and W are finite-dimensional with dimensions n and m respectively. Let $r = rank(T)$. Then $dimker(T^*) + r = m$ (by a theorem done earlier). By nullity-rank, $Rank(T^*) = r = Rank(T)$. Now if $g = T^* f$, and $v \in Ker(T)$, then $g(v) = f(T(v)) = f(0) = 0$. Hence $Range(T^*) \subset Ann(ker(T))$. But the dimension of the annihilator is $n - dim(ker(T)) = Rank(T) = Rank(T^*)$. Hence $Range(T^*) = Ann(Ker(T))$. □

In other words, $Tv = w$ iff w is annihilated by $ker(T^*)$. This sort of an observation in infinite-dimensions leads to solutions of certain PDE. We have another result.

Proposition 3.4. *Let V, W be finite-dimensional v. spaces with ordered bases $\mathcal{B}, \mathcal{B}'$. Let $T : V \to W$ be a linear map and A be the matrix relative to $\mathcal{B}, \mathcal{B}'$. Let B be the matrix of T^* relative to corresponding dual bases. Then $B = A^T$.*

Proof. $B_{ij} = T^* \vec{e^*_{j,\mathcal{B}'^*}}_{i,\mathcal{B}^*} = T^*(e^*_{j,\mathcal{B}'^*})(e_{i,\mathcal{B}}) = e^*_{j,\mathcal{B}^{*\prime}}(T(e_{i,\mathcal{B}})) = A_{ki} e^*_{j,\mathcal{B}^{*\prime}} e_{k,\mathcal{B}'} = A_{ji}$. □

Using these results, we can conclude that the row rank equals the column rank. Indeed, a matrix A defines a linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ given by $Tv = Av$. The column rank of A is $Rank(T)$. The row rank of A is $Rank(A^T) = Rank(T^*) = Rank(T)$.