NOTES FOR 19 SEPT (THURSDAY)

1. Recap

(1) Defined the double dual and proved it is isomorphic to V for finite-dimensional spaces.
(2) Defined the transpose and proved a theorem regarding it (which implied the row rank=column rank theorem).

2. Tensor products

The stress in a material in material is the force per unit area. But this quantity depends on the direction, i.e., *Stress* is a linear map that takes the normal to a plane to a force vector. Physicists call this quantity the stress tensor. For them, more or less anything that has indices and transforms the "correct" way when you change your basis is called a tensor. We want to give this concept a formal footing.

Firstly, let V, W, Z be vector spaces over a field $\mathbb{F}$. Bilinear maps are maps $T : V \times W \to Z$ satisfying $T(av_1 + bv_2, w) = aT(v_1, w) + bT(v_2, w)$ and $T(v, aw_1 + bw_2) = aT(v, w_1) + bT(v, w_2)$ - basically linear in each variable. Likewise, if V_i are vector spaces, then $T : V_1 \times V_2 \times \ldots \to Z$ is called a multilinear map if it is linear in each variable. If $Z = \mathbb{F}$, a multilinear map is called a multilinear form sometimes. Here are examples :

(1) $Tr : Mat_{n\times n}(\mathbb{F}) \times Mat_{n\times n}(\mathbb{F}) \to \mathbb{F}$ given by $Tr(A, B) = Tr(AB)$ is a bilinear form. Interestingly enough, $Tr(B, A) = Tr(BA) = Tr(AB) = Tr(A, B)$. Such a bilinear map is called a symmetric bilinear map. More generally, $T : V \times V \times V \times \ldots V \to \mathbb{F}$ is called a symmetric multilinear form if $T(v_{\sigma 1}, v_{\sigma 2}, \ldots) = T(v_1, v_2 \ldots)$ where σ is any permutation.
(2) Let $char(\mathbb{F}) \neq 2$. The map $A : \mathbb{F}^2 \times \mathbb{F}^2 \to \mathbb{F}$ given by $A((v, w), (a, b)) = vb - wa$ is a bilinear form. Moreover, $A((a, b), (v, w) = -A((v, w), (a, b))$. Such a bilinear form is called an alternating bilinear form (or sometimes simply, and confusingly, a form). More generally, $T : V \times V \times V \times \ldots V \to \mathbb{F}$ is called an alternating multilinear form if $T(v_{\sigma 1}, v_{\sigma 2}, \ldots) = (-1)^{sgn(\sigma)} T(v_1, v_2 \ldots)$ where σ is any permutation.
(3) The map $T : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ given by $T(v, w) = \sum_i v_i w_i$ is a symmetric bilinear form.
(4) $T : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by $T(v, w) = \sum_i v_i \bar{w}_i$ is NOT a bilinear form because it is not linear in the second variable. However, it is "conjugate-linear" - sometimes called sesquilinear (sesqui means one-and-a-half). We shall deal with such beasts later on.

Now that we have many examples, what can we say about the set of bilinear forms $Bilin(V \times W, \mathbb{F})$? In an obvious way, it is a vector space (this is true even for multilinear maps in general). What is the dimension of this space ? For example, the space of bilinear forms $B(\mathbb{F} \times \mathbb{F}, \mathbb{F})$ consists of $B(v, w) = vB(1, w) = vwB(1, 1)$ is 1-dimensional. More generally, $B(v, w) = B(\sum_i v_i e_i, \sum_j w_j f_j) = \sum_{i,j} v_i w_j B(e_i, f_j)$. Note that $B_{ij} : V \times W \to Z$ given by $B_{ij}(v, w) = v_i w_j$ are linearly independent (why ?) Hence the space of bilinear maps has dimension mn.

Can bilinear (or multilinear) maps be interpreted through the lens of honest linear maps ? The answer to this question leads to the construction of Tensor products. Here is a theorem.

Theorem 2.1. *Let V, W be vector spaces over $\mathbb{F}$. There exists a vector space $V \otimes W$ (called the tensor product of V and W) and a bilinear map $\pi : V \times W \to V \otimes W$ satisfying the following property (called a universal* *property) : Given any vector space Z over $\mathbb{F}$, and a bilinear map $L : V \times W \to Z$, there exists a unique linear map $\tilde{L} : V \otimes W \to Z$ such that $L = \tilde{L} \circ \pi$.*

Before we prove this theorem, note that the universal property uniquely characterises tensor products upto isomorphism. Indeed, if $(U_1, \pi_1), (U_2 = V \otimes W, \pi_2)$ are two vector spaces satisfying the universal property, then since $\pi_1 : V \times W \to U_1$ is a bilinear map, there exists a unique linear map $\tilde{\pi}_1 : U_2 \to U_1$ such that $\pi_1 = \tilde{\pi}_1 \circ \pi_2$. Likewise, there exists a unique linear map $\tilde{\pi}_2 : U_1 \to U_2$ such that $\pi_2 = \tilde{\pi}_2 \circ \pi_1$. Thus, $\tilde{\pi}_1 = \tilde{\pi}_1 \circ \tilde{\pi}_2 \circ \tilde{\pi}_1 = id \circ \tilde{\pi}_1$. But taking $Z = U_2$ and $L = id \circ \pi_2$, we see that by uniqueness of the induced linear map, $\tilde{\pi}_1 \circ \tilde{\pi}_2 = id$. Interchanging the roles of U_1, U_2 we see that $\tilde{\pi}_1 = \tilde{\pi}_2^{-1}$.
The universal property is actually quite powerful (and in principle it is enough to do anything you want with tensor products. But an explicit construction is also useful). For instance, $V \otimes W$ is naturally isomorphic to $W \otimes V$ (with the isomorphism preserving the universal property) : Indeed, consider the bilinear isomorphism $T : V \times W \to W \times V$ given by $T(v, w) = (w, v)$ and apply the universal property. Likewise (exercise) one can show that $(V \otimes W) \otimes U \equiv V \otimes (W \otimes U)$. Elements of a tensor product are called tensors. In particular, elements of $V \otimes V \otimes \ldots V (r\ times) \otimes V^* \otimes V^* \ldots (s\ times)$ are called tensors of type (r, s) (r-contravariant and s-covariant).

The universal property also helps us calculate $dim(V \otimes W)$ if V, W are finite-dimensional. Indeed, let $Z = \mathbb{F}$. Then there is a linear isomorphism (why is the correspondence linear ?) between $L(V \otimes W, \mathbb{F}) = (V \otimes W)^*$ and $Bilin(V \times W, \mathbb{F})$. We know that the latter is mn-dimensional and hence $dim(V \otimes W) = dim((V \otimes W)^*) = mn$.

Proof. Morally, we want to define $\pi(v, w) = v \otimes w$ as a symbol that is bilinear in v, w. To do this, we simply impose an equivalence relation (of bilinearity) on a huge set consisting of formal linear combinations of the type $\sum_{ij} c_{ij}(v_i, w_j)$. In more detail, define the free vector space $F(S)$ on a set over a field $\mathbb{F}$ as the direct sum $\oplus_S \mathbb{F}$. Basically, a vector in $F(S)$ is a formal linear combination $\sum_i c_i s_i$ where $c_i \in \mathbb{F}$ and $s_i \in S$. (In particular, elements of S form a basis.) In particular, consider $F(V \times W)$. Let Z be the subspace spanned by vectors of the form $(v, w) + (v', w) - (v + v', w)$, $(v, w) + (v, w') - (v, w + w')$, $c(v, w) - (cv, w)$, and $c(v, w) - (v, cw)$ for all $c \in \mathbb{F}, v, v' \in V, w, w' \in W$. We define $V \otimes W = F(V \times W)/Z$ as a quotient space. In other words, we simply set $[(v, w)] + [(v', w)] = [(v + v', w)]$, etc. Denote $[(v, w)]$ by $v \otimes w$ and define $\pi : V \times W \to V \otimes W$ as $(v, w) \to v \otimes w$. Note that $\pi(cv + c'v', w) = c\pi(v, w) + c'\pi(v', w)$ and so on. So π is a bilinear map.
(Warning : Do note that not every vector in $V \otimes W$ is of the form $v \otimes w$!! Such vectors are called "pure/indecomposable tensors". The minimum number of pure tensors needed to write an element is called the tensor rank of that element. This concept is useful in theoretical computer science. See Strassen's algorithm on wikipedia.)
(To be continued....) □