NOTES FOR 1 AUG (THURSDAY)

1. Logistics

The textbooks for this course are as follows.

(1) Linear algebra by Hoffman and Kunze (2nd Ed). This will be our main textbook.
(2) Linear algebra by S. Lang.
(3) Linear algebra and its applications by G. Strang.
(4) Algebra by M. Artin.

Grading policy : There will be Quizzes (15%), a midterm (35%) and a Final (50%). The class will be held on Tuesdays and Thursdays from 2-3:30 in LH-4. The Quizzes (roughly once every fortnight) will be held during tutorial sessions. They will contain a couple of problems from your HWs. The teaching assistants for this course are Prateek Kumar, G V K Teja, and Projesh Nath. If your roll number's last digit is $0 \ mod \ 3$, you are assigned to Prateek Kumar, if it is $1 \ mod \ 3$ then to GVK Teja and to Projesh Nath if it is $2 \ mod \ 3$.

2. Teaser-Trailer

By the end of this course I hope you will be able to answer (or at least understand the answer if you read it) the following kinds of questions.

(1) This one is easy but there is an underlying point : Let $c_1, c_2, \ldots, c_n$ be real numbers such that $c_1 + c_2 e^{ix} + c_3 e^{2ix} + \ldots = 0$ for all $x \in \mathbb{R}$. Then $c_i = 0$ for all i.
(2) Once upon a time, in the city state of Odd Town, there lived n residents. They formed clubs, subject to the following rules:
   (a) Each club consists of an odd number of residents.
   (b) The number of members common to two different clubs is always even.

   Note that the second condition allows two clubs to be disjoint (after all, zero is an even number). Note also that the two conditions together imply that two different clubs cannot have the same set of members. The question is: what is the maximum number of clubs that they could form?
(3) If A is a 3×3 matrix of real numbers satisfying $A^{10} = 0$, then prove that $A^3 = 0$.
(4) Find the n^{th} Fibonacci number.
(5) Solve $\frac{dy}{dt} = 2x + 3y$, $\frac{dx}{dt} = 3x + 2y$.
(6) The chance of it raining tomorrow if it rains today is 0.7 and if it does not rain, it is 0.5. Given that it rained today, what is the chance of it raining after many days ?
(7) The price of a house vs its area in a particular locality is sketched on a graph. What is the best straight line that fits this data ?
(8) Prove that the sum of two algebraic numbers is an algebraic number.
(9) Every finite field of characteristic p (a prime) has cardinality p^n.

3. Fields

The origin of linear algebra lies in solving systems of linear equations. Now we know that there is a formula (Cramer's rule) and many efficient algorithms (Gaussian elimination for instance) that do this job. Since studying linear equations over arbitrary fields is useful for number theory as well as algebraic geometry, we want to do everything over arbitrary fields. Recall that fields are things like $\mathbb{R}$ and $\mathbb{C}$ where we can multiply, add, subtract, and divide (by non-zero elements). The only thing we cannot do in general is to talk about order and about the size (norm) of elements. The definition of a field $(\mathbb{F}, +, \times, 0, 1)$ is as follows.

(1) $(\mathbb{F}, +, 0)$ is an Abelian group.
(2) $\mathbb{F}^* = (\mathbb{F} - \{0\}, \times, 1)$ is an Abelian group.
(3) Distributivity : $a \times (b + c) = a \times b + a \times c$.

We say that a map $f : F \to G$ is a field morphism if $f(x + y) = f(x) + f(y), f(xy) = f(x)f(y)$ and $f(1) = 1$. It is an isomorphism if it has an inverse that is also a field morphism. The characteristic of a field is the smallest non-negative integer n such that $n.1 = 1 + 1 + \ldots = 0$. If the characteristic is not finite, it is taken to be 0. Here are examples and counterexamples.

(1) $\mathbb{Z}$ is not a field because multiplicative inverses do not exist.
(2) $\mathbb{Q}$ is a field.
(3) $\mathbb{R}$ is a field (in fact it is an ordered field and the only complete ordered field upto isomorphism).
(4) $\mathbb{C}$ is a field (it is algebraically closed, i.e., every polynomial with complex coefficients has a complex root and in fact it is the algebraic closure of $\mathbb{R}$).
(5) $\mathbb{C}[x]$ is not a field.
(6) $\mathbb{Z}_n$ is a field if n is a prime and is not a field if n is not a prime.
(7) Every finite field has prime characteristic and has cardinality p^n.

4. Linear equations - Equivalent equations

Let A_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ be mn elements of a field $\mathbb{F}$ (which we shall fix from now onwards. We will largely deal with reals and the complex numbers in this course). The numbers are packaged into a matrix $[A]_{ij} = A_{ij}$. Then $x_1, \ldots, x_n \in \mathbb{F}$ are said to solve a system of linear equations with coefficients A_{ij} and right hand sides $y_1, \ldots, y_n \in \mathbb{F}$ if $\sum_j A_{ij}x_j = y_i \ \forall \ i$. A linear combination of equations is defined to be $\sum_{i,j} c_i A_{ij} x_j = \sum c_i y_i$ where c_i are field elements. Two systems of equations are said to be equivalent if each equation in each system is a linear combination of the equations from the other system. It is trivial to see that equivalent systems of linear equations have the same solutions. We will abbreviate our system of equations as $AX = Y$ where X, Y are column vectors/matrices.

5. Row operations

The point is to solve linear equations by adding and subtracting equations to isolate at least some variables, and then back-substitute the values found to get the values of the other variables. To do all of this, we only need to keep track of the coefficients, i.e., the matrix A. To this end, we define the following elementary row operations on an $m \times n$ matrix A over a field $\mathbb{F}$.

(1) Multiplication of a row by an element $c \in \mathbb{F}$. (Elements of the underlying field are called scalars.) That is $A_{rj} \to cA_{rj} \ \forall \ 1 \leq j \neq n$
(2) Exchanging rows, i.e., $A_{rj} \to A_{sj}$ and $A_{sj} \to A_{rj}$.
(3) Replacing the r^{th} row of A by the r^{th} row plus c times the s^{th} row (where $r \neq s$), i.e., $A_{rj} \to A_{rj} + cA_{sj}$.

The first observation is : Every elementary row operation is invertible by a elementary row operation of the same type.

We say that A and B are row-equivalent if B can be obtained from A by a sequence of elementary row operations.

Proposition 5.1. *If A and B are row equivalent matrices, then $AX = 0$, $BX = 0$ have the same solutions.*

One can prove the proposition inductively by noting an elementary row operation is simply a linear combination of equations (and linear combinations do not change solutions) and that an elementary row operation is inverted by an elementary row operation (in fact of the same type). It is easy to see examples of row operations being applied to produce solutions of linear equations.