NOTES FOR 20 AUG (TUESDAY)

1. Recap

(1) Gave examples and non-examples of vector spaces. (Defined abstract polynomials and polynomial functions in the process.)
(2) Defined subspaces, and subspaces spanned by sets. Gave examples and non-examples.

2. Bases and dimension

Def : Let V be a vector space over $\mathbb{F}$. A subset $S \subset V$ is said to be linearly independent if no non-trivial linear combination from S is 0. (Otherwise it is said to be linearly dependent.) Note that if 0 is in a set, then $1.0 = 0$ and hence linear dependence holds.

Def : A (Hamel) basis is a linearly independent set whose span is V. V is said to be finite-dimensional if it has a finite basis. Examples and non-examples :

(1) $(1, 2, 0)$ and $(0, 2, 1)$ are linearly independent over $\mathbb{Z}_3$.
(2) $\mathbb{F}^n$ is finite-dimensional because e_i is a finite basis.
(3) $M_{m\times n}(\mathbb{F})$ is finite-dimensional because $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$ forms a basis of size mn.
(4) The space of all polynomials is infinite-dimensional. Indeed, if there is a finite basis, let m be the maximum degree. Polynomials of degree $> m$ are not linear combinations.
(5) Caution : $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ is not a finite linear combination. In the definition of a basis, only finite linear combinations are considered. For countable linear combinations, there is another concept called a Schauder basis which is not as general as the usual Hamel basis.
(6) $\mathbb{R}$ over $\mathbb{Q}$ is not finite-dimensional. Indeed, if it is, $x = \sum_{i=1}^{n} a_i x_i$ where $x, x_i \in \mathbb{R}$ and $a_i \in \mathbb{Q}$. This means that $\mathbb{R}$ is countable - a contradiction. (So in fact, $\mathbb{R}$ over $\mathbb{Q}$ is not even countably infinite-dimensional.)
(7) Let P be an invertible $n \times n$ matrix over a field $\mathbb{F}$. Then the columns P_i of P are linearly independent. Indeed, if not, then $\sum_i c_i P_i = 0$ which means that $Pc = 0$ (remember that if $C = AB$, then the rows of C are linear combinations of those of B whereas the columns of C are linear combinations of the columns of A) and hence $c = 0$. In fact, the columns form a basis for $\mathbb{F}^n$. Indeed, if $y \in \mathbb{F}^n$, there exists a c so that $Pc = y$ which means that y is a linear combination of the columns of P.
(8) Let A be an $m \times n$ matrix. The set of solutions of $AX = 0$ form a subspace of $\mathbb{F}^n$. Row reduce A to its row echelon form R. Let the pivots occur at positions $k_1, k_2, \ldots, k_r$ where r is the row rank. Then $x_{k_i} = -\sum_{j>k_i} R_{ij} x_j$. In particular, $x_{k_r} = -\sum_{j>k_r} R_{rj} x_j$. Likewise, all the pivoted variables are determined and the rest are undetermined. We claim that the vectors $\tilde{e}_j$ obtained by setting one of the free variables to 1 and the rest to zero form a basis for the solution space. Indeed, they are linearly independent and every vector in the solution space is a linear combination.

We prove now the notion of dimension is well-defined.

Theorem 2.1. *Let V be a vector space which is spanned by a finite set $\beta_1, \ldots, \beta_m$. Then any linearly independent set of vectors in V is finite and contains no more than m elements.*

Proof. Let $S \subset V$ be linearly independent containing $s_1, \ldots, s_{m+1}$. Then $s_i = \sum_j C_{ji}\beta_j$ where C is an $m \times (m+1)$ matrix. In this situation (using the row echelon form) we see that there is a non-trivial solution to $CX = 0$. Hence $\sum_i x_i s_i = \sum_{i,j} \beta_j (Cx)_j = 0$. Thus we are done. $\square$

As a corollary, if V is a finite-dimensional vector space, every basis has the same number of elements that we shall call the dimension of V. For instance, $\mathbb{F}^n$ has dimension n. The space of $m \times n$ matrices has dimension mn. The 0 vector space has dimension 0 (by convention).
To state Zorn's lemma, we need the notion of a partially ordered set (poset)
$S, \leq$: It is a set S with a partially defined relation $\leq$. It satisfies the property that $a \leq b$ and $b \leq a$ iff $a = b$. Moreover, $a \leq a$, $a \leq b$, $b \leq c$ implies that $a \leq c$. So it is reflexive, antisymmetric, and transitive. However, not every two elements may be comparable (that is why the term "partial"). A partially ordered set where any two elements are comparable is called a totally ordered set or a chain.

Here are examples and non-examples,

(1) $=$ is a partial order on any set.
(2) $\leq$ on $\mathbb{R}$ is a total order.
(3) $A \subset B$ is a partial order (but not a total order) on $\mathcal{P}(X)$. Likewise, subspaces of vector spaces under inclusion. Also, the subsets of linearly independent elements of a vector space under inclusion.
(4) The set of events in special relativity where $X \leq Y$ if Y is in the future light cone of X.
(5) $\neq$ is not a partial order.

Here is are a few more definitions : A maximal element in a poset is an element a such that there exists no other element b satisfying $a \leq b$. A maximum element is an element such that every element is $\leq$ to it. (It is clear by induction that in a totally ordered set, every finite subset has a maximum element in it.) An upper bound of a subset A of a poset P is an element $x \in P$ such that $x \geq a \; \forall \; a \in A$.
Zorn's lemma : A poset in which every chain (that is, a totally ordered subset) has an upper bound has a maximal element.