NOTES FOR 22 AUG (THURSDAY)

1. Recap

(1) Defined bases and gave examples.
(2) Proved that the notion of dimension makes sense.
(3) Defined posets, upper bounds, maximal elements, and chains and stated Zorn's lemma.

2. Bases and dimension

Theorem 2.1. *Every vector space has a basis.*

Proof. Consider $\mathcal{B} \subset \mathcal{P}(V)$ consisting of sets of linearly independent elements under inclusion. Given a chain $C \subset \mathcal{B}$, consider $M = \cup_{A \in C} A$. M is clearly an upper bound of the chain. We claim that $M \subset \mathcal{B}$. Indeed, if not, then there exists a linearly dependent collection of vectors $v_1, \ldots, v_n$ in M. Since each $v_i \in A_i$ for some i, let A_k be a maximum of these finitely many elements in the totally ordered subset C. Hence $v_1, \ldots, v_n \in A_k$ and this is a contradiction. Therefore by Zorn's lemma, there exists a maximal linearly independent set I of vectors. If this set does not span V, then there is an element $v \in V$ such that $J = \{v\} \cup I \subset \mathcal{B}$. Here we use the following lemma.

Lemma 2.2. *Let $S \subset V$ be a linearly independent subset. Suppose $\beta \in V \cap W_S^c$. Then $S' = S \cup \{\beta\}$ is linearly independent.*

Proof. Indeed, suppose $c\beta + \sum_i c_i s_i = 0$, then if $c = 0$, we have a contradiction unless $c_i = 0 \ \forall\ i$. If $c \neq 0$, then also we have a contradiction because β is a linear combination of elements in S. □

But this expression contradicts that the fact that I is maximal. Hence I forms a basis of V. □

Now we prove

Proposition 2.3. *Let $W \subset V$ be a subspace of a possibly infinite-dimensional vector space V. Then every linearly independent subset S of W is a part of a basis of W. Moreover, if V is finite-dimensional, then such an S is finite and W is also finite-dimensional.*

Proof. Consider the set $\mathcal{S} \subset P(W)$ of all linearly independent subsets of W containing S and partially order this set $\mathcal{S}$ by inclusion. Let $\mathcal{A}$ be a chain. Then $B = \cup_{A \in \mathcal{A}} A$ is an upper bound because $S \subset B$ and as above, B is linearly independent. Hence, there is a maximal element in $\mathcal{S}$. As above, this maximal element is a a basis.
If V is finite-dimensional, since $S \subset V$, S is finite. Hence any basis of W - which is of course a linearly independent set, is finite. □

We have the following corollaries.

(1) If $W \subset V$ is a proper subspace and V is f.d, then so is W and $dim(W) < dim(V)$: Indeed, taking a basis B of W, it can be extended to a basis of V and hence $dim(W) \leq dim(V)$. Since there exists a vector $v \in V \cap W^c$, $dim(W) < dim(V)$.
(2) In every vector space V, every non-empty linearly independent set of vectors is a part of a basis.

(3) Let A be an $n \times n$ matrix over a field. Suppose the rows of A form a linearly independent set. Then A is invertible : Row operations on A create matrices with linearly independent rows. (An easily verified fact.) Therefore, the row echelon form of A has no zero rows. Thus it is the identity and A is invertible.

We have the following reasonable sounding theorem.

Theorem 2.4. *If W_1, W_2 are finite-dimensional subspaces of a vector space V, then so is $W_1 + W_2$ and $dim(W_1) + dim(W_2) = dim(W_1 \cap W_2) + dim(W_1 + W_2)$.*

Proof. Let $p_1, \ldots, p_k$ be a basis of $W_1 \cap W_2$. Extend this basis to $q_{k+1}, \ldots, q_m$ of W_1 and $r_{k+1}, \ldots, r_n$ of W_2. Then consider the subspace W spanned by the linearly independent set $U = \{p_1, \ldots, p_k, q_{k+1}, \ldots, q_m, r_{k+1}, \ldots, r_n\}$. (Why is U linearly independent ?) Clearly, $W_1 + W_2$ is spanned by U. Hence we are done. □

Warning : Another reasonable sounding statement $dim(W_1) + dim(W_2) + dim(W_3) = \ldots$ (fill in the blanks) is false !!!

Def : An ordered basis $\mathcal{B}$ of an n-dimensional vector space V is a sequence of vectors $e_1, \ldots, e_n$ that form a basis for V.

Given an ordered basis of a finite-dimensional vector space V, every vector v can be uniquely identified as an element of $\mathbb{F}^n$, i.e., the map $(v_1, \ldots, v_n) \to \sum_i v_i e_i$ is a bijection. Moreover, this bijection preserves the vector space structure. Indeed, this situation motivates the following definitions :

Def : A linear transformation $T : V \to W$ between vector spaces V and W over the same field is a function such that $T(av + bw) = aT(v) + bT(w)$. A linear isomorphism is a bijective linear transformation. Note that the inverse of a linear isomorphism is also a linear transformation. Indeed, $T(aT^{-1}v + bT^{-1}w) = aT(T^{-1}v) + bT(T^{-1}w) = av + bw$ and hence $aT^{-1}v + bT^{-1}w = T^{-1}(av + bw)$. Two vector spaces are said to be isomorphic if there is a linear isomorphism between them.

We shall return to linear transformations later. For now, note that $\mathbb{F}^n$ is isomorphic to V by the linear transformation given above.

However, it is best to not fix a basis for a vector space. The components/coordinates in a particular basis are usually written as column vectors $\vec{v}_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Some people like writing the basis vectors themselves in a row as $e^T = [e_1\ e_2\ e_3\ \ldots\ e_n]$ so that conveniently, $v = e^T \vec{v}_{\mathcal{B}}$ (as matrix multiplication). Clearly, if we change a basis, the components will change.

Theorem 2.5. *Let V be an n-dim vector space and let $\mathcal{B}, \mathcal{B}'$ be two ordered bases. Then there is a unique invertible $n \times n$ matrix P such that $\vec{v}_{\mathcal{B}} = P\vec{v}_{\mathcal{B}'}$ and $\vec{v}_{\mathcal{B}'} = P^{-1}\vec{v}_{\mathcal{B}}$ for all $v \in V$. The columns of P are given by $P_i = \vec{e'_i}_{\mathcal{B}}$.*

Conversely, given an ordered basis $\mathcal{B}$ and an invertible $n \times n$ matrix P, there is a unique ordered basis $\mathcal{B}'$ such that $\vec{v}_{\mathcal{B}} = P\vec{v}_{\mathcal{B}'}$ and $\vec{v}_{\mathcal{B}'} = P^{-1}\vec{v}_{\mathcal{B}}$ for all $v \in V$.