NOTES FOR 22 OCT (TUESDAY)

1. Recap

(1) Stated the JCF theorem.
(2) Defined generalised eigenvectors and generalised eigenvalues. Proved a couple of properties.
(3) Reduced the proof of JCF to nilpotent operators.

2. The Jordan Canonical Form

Theorem 2.1. *Let $T : V \to V$ be an operator between finite-dimensional vector spaces. Assume that all the eigenvalues of T are in $\mathbb{F}$. Then there exist unique invariant subspaces V_i such that $V = \oplus_i V_i$ and there is a basis of V obtained through bases of V_i such that T is of the Jordan Canonical Form in this basis, i.e., T is upper-triangular, with the only possible super-diagonal elements being 1. The V_i are direct sums of Jordan blocks where a Jordan block looks like* $\begin{bmatrix} \lambda & 1 & 0 & \dots \\ 0 & \lambda & 1 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda \end{bmatrix}$*. (The Jordan Blocks are unique up to permutation.)*

Proof. Assume from now on that T is nilpotent. Once again, let $K = Ker(T)$ and $R = Ran(T)$. By induction, there exists a linearly independent set $w_1, \dots, w_r \in R$ such that the corresponding Jordan chains span R and have exponents/orders d_i. Let $v_k \in V$ such that $Tv_k = w_k$. We claim that

(1) The Jordan chains of v_k are linearly independent : Indeed, if $\sum_k c_k v_k + c_{Tk} T v_k + \dots = 0$, then $\sum_k c_k w_k + c_{Tk} T w_k + \dots = 0$. Thus, all the c_j are zero except for the ones corresponding to $T^{d_i} v_i = T^{d_i - 1} w_i$. But a linear combinations of some Jordan chain vectors of w_k is 0 implies that the coefficients are 0.
(2) There are linearly independent vectors $e_1 \dots, e_k \in K$ such that they along with Jordan chains of v_k form a basis of V : Firstly, $K + U = V$ where U is the span of the Jordan chains of v_k. Indeed, if $v \in V$, then $Tv = R = TU = Tu$ where $u \in U$. Hence $v - u \in K$. Extend the Jordan chain basis of U to any basis of V by adding $e_i \in K$. We are done with proving existence.

For uniqueness, note that we can again restrict ourselves to nilpotent operators. We can use the following "algorithm" (not quite ! We don't know how to put it on a computer !) to calculate the Jordan form is :

(1) Compute the eigenvalues. Calculate the minimal polynomial. Its multiplicities k give the maximum sizes of the Jordan blocks.
(2) Compute the dimensions k_i of the kernels $K_r = N((T - \lambda I)^r)$ for all $r \leq k_\lambda$.
(3) $k_i - k_{i-1}$ is the number of blocks whose size is at least i.

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As a corollary,

Corollary 2.2. *TFAE.*

(1) *T is diagonalisable.*

(2) *Every generalised eigenvector is an eigenvector.*
(3) *All the Jordan blocks are* 1×1 *blocks.*
(4) *The minimal polynomial has no repeated roots.*

Proof. $1 \Rightarrow 2$: If $(T - \lambda_i)^n v = 0$, then writing $v = \sum_k c_k e_k$ where e_i are eigenvectors, $\sum_{k|\lambda_k \neq \lambda_i} c_k(\lambda_k - \lambda_i)^n e_k = 0$. Hence, all such $c_k = 0$. So v is in the eigenspace of λ_i.
$2 \Rightarrow 3$: If there is a non-1×1 block, it is easy to see that its generator is a non-eigenvector but is a generalised one.
$3 \Rightarrow 1$: Trivial. $1 \Rightarrow 4$ was done earlier. For $4 \Rightarrow 1$, if there is a generalised eigenvector v of λ that is not an eigenvector, then $m(T)v \neq 0$ as can be seen by an earlier computation to find the minimal polynomial of a Jordan block. □

Here is another consequence : Let $T : V \to V$ be an operator over a f.d. vector space over a characteristic 0 field satisfying $T^r = 1$ for some r. Assume that $t^r - 1 = 0$ has all its roots (with multiplicity) in the field. Then T is diagonalisable.
Indeed, firstly, the eigenvalues are some roots of $t^r - 1 = 0$. Secondly, if $(T - \lambda I)^2 v = 0$ and $v \neq 0$, then let $w = (T - \lambda I)v$. Hence, $Tw = \lambda w$. So $0 = (T^r - I)v = (T - \lambda I)(T^{r-1} + \ldots + \lambda^{r-1})v = 0$ which means that $0 = (T^{r-1} + \ldots)w = r\lambda^{r-1}w$ and hence $w = 0$.
Here are a couple of examples (from Artin). Computing the change of basis is even more painful. (One needs to not only calculate dimensions of K_r but also bases such that a few are Jordan chains generated by elements in $K_k - K_{k-1}$, etc.)

(1) $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. $p_A(t) = t^3$. It turns out that $A^2 \neq 0$. So, if v is any vector such that $A^2 v \neq 0$, $v, Av, A^2 v$ will be a Jordan chain. A is similar to just one Jordan block.
(2) $B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix}$. $p_B(t) = t^3$. Actually, $B^2 = 0$ is the minimal polynomial. So there is a 2×2 Jordan block and a 1×1 block. Let v be a vector such that $Bv \neq 0$. Then, v, Bv generate the 2×2 block. An eigenvector e linearly independent of Bv generates the other block.

3. Inner products

The proof that e^{ikx} are linearly independent was by multiplying with e^{-ilx} and integrating. All the integrals except one became zero. This is reminiscent of taking dot products in $\mathbb{R}^n$. So we generalise this concept by means of a definition : Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ from now onwards till the end of time (unless specified otherwise). Let V be a vector space over $\mathbb{F}$. A map $\langle , \rangle : V \times V \to \mathbb{F}$ is said to be an inner product if the following hold.

(1) $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$.
(2) $\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$.
(3) $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
(4) $\langle v, v \rangle \geq 0$ with equality holding iff $v = 0$.

Note that the last two properties do not make sense for most other fields. Basically, an inner product is a sesquilinear form that satisfies a positivity property.