NOTES FOR 27 AUG (TUESDAY)

1. Recap

(1) Proved that every vector space has a basis. In fact, that every linearly independent subset can be extended to a basis.
(2) $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$ but the analogous statement for three subspaces is false ! (take three lines in $\mathbb{R}^2$ for instance).
(3) Defined ordered bases, linear maps, and components.

2. Bases and dimension

Theorem 2.1. *Let V be an n-dim vector space and let $\mathcal{B}, \mathcal{B}'$ be two ordered bases. Then there is a unique invertible $n \times n$ matrix P such that $\vec{v}_{\mathcal{B}} = P\vec{v}_{\mathcal{B}'}$ and $\vec{v}_{\mathcal{B}'} = P^{-1}\vec{v}_{\mathcal{B}}$ for all $v \in V$. The columns of P are given by $P_i = \vec{e'}_{i\mathcal{B}}$.*
Conversely, given an ordered basis $\mathcal{B}$ and an invertible $n \times n$ matrix P, there is a unique ordered basis $\mathcal{B}'$ such that $\vec{v}_{\mathcal{B}} = P\vec{v}_{\mathcal{B}'}$ and $\vec{v}_{\mathcal{B}'} = P^{-1}\vec{v}_{\mathcal{B}}$ for all $v \in V$.

Proof. The first direction : Indeed, there are unique scalars P_{ij} such that $e'_j = \sum_i P_{ij} e_i$. If $v = \sum_j v'_j e'_j = \sum_i v_i e_i$, then $\sum_j v'_j \sum_i P_{ij} e_i = \sum_i v_i e_i$. Hence, by uniqueness of the coordinates/components, $\sum_j P_{ij} v'_j = v_i$, i.e., $\vec{v}_{\mathcal{B}} = P\vec{v}_{\mathcal{B}'}$. P is invertible by interchanging the roles of e' and e. Now such a P is unique because P_i is P acting on the i^{th} standard basis vector of $\mathbb{F}^n$ which equals $\vec{e'}_{i\mathcal{B}}$.
The converse : If there is such an ordered basis, then clearly, $e'_j = \sum_i P_{ij} e_i$. So uniqueness holds. Now we only have to prove that e'_j do form a basis. Indeed, if $\sum_j c_j e'_j = 0$, then $\sum_j c_j \sum_i P_{ij} e_i = 0$ or $Pc = 0$ and hence $c = 0$. So they are n linearly independent vectors and hence form a basis. □

Here are examples.
(1) The standard coordinate basis in $\mathbb{F}^n$.
(2) Let $P = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. It is clearly invertible and hence can be used to define another basis. Note that it corresponds to rotating the standard basis by θ. The old components are related to the new ones by P.
(3) The Hermite polynomials $h_1 = 1, h_2 = x, h_3 = x^2 - 1$ also form a basis in the space of polynomials of degree ≤ 2. Hence, $P_1 = P([1,0,0]_{\mathcal{H}}) = [1,0,0]^T$, $P_2 = P([0,1,0]_{\mathcal{H}}) = [0,1,0]^T$, $P_3 = P([0,0,1]_{\mathcal{H}}) = [-1,0,1]^T$.

3. Back to matrices

Let A be an $m \times n$ matrix over a field. The subspace of $\mathbb{F}^n$ spanned by the rows of A is called the row space of A. (Likewise the column space is a subspace of $\mathbb{F}^m$.) If $B = PA$, then the rows of B are linear combinations of the rows of A and hence the row space of B is a subspace of the row space of A. If P is invertible, then $P^{-1}B = A$ and hence the row spaces of A and B coincide. In particular, the row space of A coincides with that of its row echelon form.

Proposition 3.1. *Let R be a non-zero row echelon matrix. Then the non-zero rows of R form a basis for the row space of R.*

Proof. By definition of row echelon matrices, the non-zero rows clearly span the row space. They are linearly independent. Indeed, if $\sum_i c_i \, row_i = 0$, then looking at the pivots, $c_i = 0$ for all i. □

As a consequence, the row rank is simply the dimension of the row space of a matrix. Now we prove an important result.

Proposition 3.2. *Elementary row operations on a column echelon $m \times n$ matrix A do not change its column rank. (Sorry, the statement in the class was incorrect.)*

Proof. Let $A_1, \ldots, A_k$ be the basis of non-zero columns for the column space of A. Firstly, note that the elementary row operations keep the zero columns as they are. So the column space is still spanned by the transformed versions of $A_1, \ldots, A_k$. If we prove that they continue to remain linearly independent, we are done.

(1) $R_i \to cR_i$ where $c \neq 0$: If i is not a pivotal row, then clearly the column rank does not change. Indeed, the pivots remain the same and hence the new set of non-zero columns are still linearly independent. If i is a pivotal row, then again the new pivot may not be 1 but is still non-zero and hence the column rank remains the same.
(2) $R_i \to R_i + cR_j$: If $c = 0$ nothing happens. So assume $c \neq 0$. If i is not a pivotal row, then the rows containing the pivots remain unchanged and hence the columns are still linearly independent. If i is a pivotal row, then if j is a pivotal row, then A_i's pivot remains unchanged. It is easy to see that the column rank remains the same (by doing the column operation $C_j \to C_j - \frac{1}{c}C_i$). If j is not a pivotal row, then if $\sum_i u_a A'_a = 0$, looking at the old pivots in A'_a for $a \neq i$ we see that $u_a = 0 \; \forall \, a$. Hence the column rank remains the same.
(3) $R_i \leftrightarrow R_j$: Exercise.

□

Before we move on, here is an example that illustrates how one goes about studying linear independence and the other concepts in $\mathbb{F}^n$: Let $V = \mathbb{R}^4$. Consider the vectors $v_1 = (1, 2, 2, 1), v_2 = (0, 2, 0, 1), v_3 = (-2, 0, -4, 3)$ and let W be the subspace spanned by them.

(1) Prove that v_1, v_2, v_3 form a basis for W.
Ans: Consider the matrix A with rows as the v_i. Reducing A to its row echelon form, we see that the row rank is 3 and hence v_i form a basis for the row space.
(2) Let $\beta = (b_1, b_2, b_3, b_4) \in W$. What are the components of β relative to the basis v_1, v_2, v_3 ?
Ans: β is in the row space. So β^T is in the column space of A^T. Hence, $\beta^T = A^T c$ where c is a column vector of coefficients. We can solve this problem by Gaussian elimination (bringing A^T to its row echelon form) to get $c^T = (b_1 - \frac{1}{3}b_2 + \frac{2}{3}b_4, -b_1 + \frac{5}{6}b_2 - \frac{2}{3}b_4, -\frac{1}{6}b_2 + \frac{1}{3}b_4)$.
(3) Let $v'_1 = (1, 0, 2, 0), v'_2 = (0, 2, 0, 1), v'_3 = (0, 0, 0, 3)$. Prove that these vectors also form a basis of W.
Ans: Again the Row echelon form comes to our rescue.
(4) Find the matrix P such that $\vec{v}_{\mathcal{B}} = P\vec{v}_{\mathcal{B}'}$.
Ans: The columns of P are $P_i = [\vec{e}'_i]_{\mathcal{B}}$.