NOTES FOR 29 OCT (TUESDAY)

1. Recap

(1) Proved the JCF theorem and proved a consequence about diagonalisability. Did a couple of examples.
(2) Proved that over fields of characteristic 0, T satisfying $T^r = I$ is diagonalisable.

2. Inner products

Here are examples and non-examples.

(1) On $\mathbb{R}^n$, the standard inner product (called the "Euclidean metric") sometimes is $\langle v, w\rangle = \sum_i v_i w_i$. Likewise, on $\mathbb{C}^n$, $\langle v, w\rangle = \sum_i v_i \bar{w}_i$. On $\mathbb{R}^n$, $\langle v, w\rangle = \sum_i c_i^2 v_i w_i$ is also an inner product if $c_i > 0$, $\forall\, i$. Note that $\langle v, w\rangle = v^T \bar{w}$ if v, w are treated as column vectors.
(2) On $\mathbb{C}^n$, the map $(v, w) = \sum_i v_i w_i$ is NOT an innner product. (It is bilinear, not sesquilinear and does not obey the positivity property.)
(3) On $\mathbb{R}^2$, $\langle v, w\rangle = v_1 w_1 - v_2 w_1 - v_1 w_2 + 4 v_2 w_2$ is an inner product. Indeed, it is clearly bilinear. Now $\langle v, v\rangle = v_1^2 - 2v_1 v_2 + 4v_2^2 = (v_1 - v_2)^2 + 3v_2^2 \geq 0$ with equality iff $v_2 = 0 = v_1$.
(4) On $Mat_{n\times n}(\mathbb{C})$, the map $\langle A, B\rangle = tr(AB^\dagger)$ is an inner product, where $B^\dagger = \overline{B^T}$. Indeed, it is sesquilinear. $tr(AA^\dagger) = \sum_i |A_{ij}|^2 \geq 0$ with equality iff $A = 0$.
(5) On the space of continuous functions $f : [0, 1] \to \mathbb{R}$, the map $\langle f, g\rangle = \int_0^1 fg dx$ is an inner product. (This inner product is called the L^2 inner product on continuous functions.)
(6) On the space of continuously differentiable functions $f : [0, 1] \to \mathbb{R}$, the map $\langle f, g\rangle = \int_0^1 fg dx + \int_0^1 f' g' dx$ is an inner product. (It is called the Sobolev $W^{1,2}$ inner product on C^1 functions.)
(7) Here is a whole class of examples : Let $T : V \to W$ be a $1-1$ linear map and let $g_W(,)$ be an inner product on W. Then clearly $g_V(x, y) = g_W(Tx, Ty)$ is an inner product on V. For instance, if A is an invertible $n \times n$ matrix, then $\langle v, w\rangle = (Aw)^\dagger Av$ is an inner product. Likewise, $(f, g) = \int_0^1 t^2 f\bar{g} dt$ is an inner product on continuous functions $f : [0, 1] \to \mathbb{C}$.

Here is an important definition : A norm $\|.\| : V \to \mathbb{R}$ is a map such that

(1) $\|\lambda v\| = |\lambda|\|v\|$.
(2) $\|v + w\| \leq \|v\| + \|w\|$. (The triangle inequality.)
(3) $\|v\| \geq 0$ with equality iff $v = 0$. (If this property is not met, it is called a seminorm.)

Given an inner product $\langle, \rangle$, here is a norm : $\|v\| = \sqrt{\langle v, \rangle v}$. Here is a proof of the triangle inequality : $\|v + w\| = \sqrt{\|v\|^2 + \|w\|^2 + \langle v, w\rangle + \langle w, v\rangle}$. At this juncture, we need the very very useful Cauchy-Schwarz inequality $|\langle v, w\rangle| \leq \|v\|\|w\|$ with equality holding iff v, w are linearly dependent. Given the CS inequality, $\|v + w\|^2 \leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\| = (\|v\| + \|w\|)^2$. So we just need to prove the CS inequality. Here is the proof :
Let $w \neq 0$ without loss of generality. Consider $f(t) = \langle v + tw, v + tw\rangle = \|v\|^2 + t^2\|w\|^2 + t(\langle v, w\rangle + \langle w, v\rangle)$. Clearly, $f(t) \geq 0$ with equality iff $v = -tw$, i.e., v, w are linearly dependent. The function $f(t)$ is minimised when $f'(t) = 0$, i.e., when $t = -\frac{\langle v,w\rangle + \langle w,v\rangle}{2\|w\|^2}$. Substituting this value of t in $f(t) \geq 0$, we get

the CS inequality.
In particular, $|\int_0^1 f\bar{g}|^2 \leq \int_0^1 |f|^2 \int_0^1 |g|^2$. Here are examples and non-examples of norms.

(1) On $\mathbb{R}^2$, $f(v) = |v_1|^2$ is not a norm. (It is a seminorm.)
(2) On $\mathbb{R}^2$, $||v|| = |v_1| + |v_2|$ is a norm (called the l^1 norm). It does not arise out of an inner product. Indeed, $(1,1) = 2 = |(1,0)| + |(0,1)|$. So v, w are not linearly dependent and yet equality holds in the triangle inequality.

When does a norm arise out of an inner product ? For starters, $\langle v+w, v+w\rangle + \langle v-w, v-w\rangle = 2||v||^2 + 2||w||^2$. This identity is called the polarisation identity. It is a necessary condition for a norm to arise out of an inner product. It is also sufficient ! (Hint: Use a linear combination of things like $||v+cw||^2$ where c is some complex number.)

Given an inner product (sometimes also called a metric) $\langle, \rangle$, and a basis e_i, we see that $\langle v, w\rangle = \sum_i v_i \bar{w}_j \langle e_i, e_j\rangle$. Define the matrix $H_{ij} = \langle e_i, e_j\rangle$. Then $\langle v, w\rangle = v^T H \bar{w} = w^\dagger \bar{H} v$. Note that, $H_{ij} = \bar{H}_{ji}$, i.e., $H^\dagger = H$. A matrix satisfying this property is said to be Hermitian. (A matrix satisfying $A^T = A$ is said to be symmetric.) Moreover, $v^\dagger H v \geq 0$ for all v with equality holding iff $v = 0$. A Hermitian matrix satisfying this condition is called positive-definite. (If the equality condition is dropped, then it is said to be positive-semidefinite.) Conversely, given a positive-definite matrix H and a basis e_i, the map $\langle v, w\rangle = v^T H \bar{w}$ defines an inner product. Here are examples and non-examples of Hermitian positive-definite matrices.

(1) Let A be an invertible matrix. Then $H = A^\dagger A$ is Hermitian (clearly) and positive-definite. Indeed, $v^\dagger A^\dagger A v = ||Av||^2_{standard} \geq 0$ with equality iff $Av = 0 \Leftrightarrow v = 0$. If A is not invertible, it is positive-semidefinite.