NOTES FOR 31 OCT (THURSDAY)

1. Recap

(1) Examples of inner products. Defined norms and gave examples/non-examples.
(2) Discussed when norms arise out of inner products (polarisation identity). Indeed, for real vector spaces, $\langle v, w\rangle := \frac{||v+w||^2 - ||v-w||^2}{4}$ is an inner product. It is easy to see that it is positive-definite and Hermitian. The bilinearity can be proved using the polarisation identity. Indeed,
   (a) $\langle v, w_1 + w_2\rangle = \langle v, w_1\rangle + \langle v, w_2\rangle$: It is easy to prove that $\langle v, w_1 + w_2\rangle = 2\langle v, w_1\rangle - \langle v, w_1 - w_2\rangle$. Indeed,

$$4\langle v, w_1 + w_2\rangle = ||v + w_1 + w_2||^2 - ||v - w_1 - w_2||^2$$
$$= 2(||v + w_1||^2 + ||w_2||^2) - ||v + w_1 - w_2||^2 - 2(||v - w_1||^2 + ||w_2||^2) + ||v - w_1 + w_2||^2$$
$$= 8\langle v, w_1\rangle - 4\langle v, w_1 - w_2\rangle. \tag{1.1}$$

   Applying it to w_2, the same expression equals $2\langle v, w_2\rangle - \langle v, w_2 - w_1\rangle$. Adding the expressions, we get the result.
   (b) $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$: Firstly, we prove that if $||v_i - v|| \to 0$, then $||v_i|| \to ||v||$. Indeed, $||v|| - ||v_i|| \leq ||v - v_i||$ and likewise $||v_i|| - ||v|| \leq ||v_i - v|| = ||v - v_i||$. Therefore, if $r \in \mathbb{R}$ and $q_i \in \mathbb{Q}$ such that $q_i \to r$, then $||v + q_i w - (v + rw)|| = |q_i - r|||w|| \to 0$ and hence $||v + q_i w||^2 \to ||v + rw||^2$. Hence, it is enough to prove this property for rational λ. Firstly, it is trivial to see that $\langle -v, w\rangle = -\langle v, w\rangle$. Hence, it is enough to prove it for positive rationals $\frac{p}{q}$. Suppose we prove it for natural numbers, then $\langle q\frac{v}{q}, w\rangle = q\langle \frac{v}{q}, w\rangle$. Hence, $\langle \frac{p}{q}v, w\rangle = \frac{p}{q}\langle v, w\rangle$. To prove it for naturals, we can easily induct on λ.
(3) Looked at inner products in bases. Defined Hermitian and positive-definite matrices.

2. Inner products

(1) The matrix $A = \begin{bmatrix} -2 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{bmatrix}$ is not positive-definite. (Why?) More generally, if $A = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$ is positive-definite (where a, c are real), then $\alpha = a|v_1|^2 + c|v_2|^2 + \bar{v}_1 b v_2 + \bar{v}_2 \bar{b} v_1 \geq 0 \; \forall \; v \neq 0$. Thus, $a, c > 0$. Moreover, $\alpha = |\sqrt{a}v_1 + \frac{b}{\sqrt{a}}v_2|^2 + |v_2|^2(-\frac{|b|^2}{a} + c) \geq 0$ iff $ac - b^2 > 0$. That is, $tr(A) > 0, \det(A) > 0$. Actually these two conditions are sufficient. (Why ?)

A related definition is this : An operator $T : V \to V$ is called Hermitian if $\langle v, Tw\rangle = \langle Tv, w\rangle$. Let e_i be a basis and T be the matrix of T. Then, $(Tw)^\dagger Hv = w^\dagger HTv$. Thus, $w^\dagger T^\dagger Hv = w^\dagger HTv$. So, for instance, if $V = \mathbb{C}^n$ and $\langle , \rangle$ is the standard inner product, then $T^\dagger = T$ as a matrix.
We have a beautiful theorem at this point (the uncertainty principle).

Theorem 2.1. *Let V be a finite-dimensional complex vector space endowed with an inner product $\langle , \rangle$. Let $A, B : V \to V$ be Hermitian operators and let $[A, B] = AB - BA : V \to V$ be their commutator. For any Hermitian operator T and a vector $\psi \in V$, define the expectation value $E_\psi(T) = (\psi, T\psi)$ and the standard deviation $\Delta_\psi T = \sqrt{E((T - E(T)I)^2)}$. Then, $\Delta_\psi A \Delta_\psi B \geq \frac{1}{2}|E([A, B])|$.*

This theorem is much more subtle in infinite-dimensions (and not true in general !) A particular infinite-dimensional version (where A corresponds to multiplication by x and B to $\frac{d}{dx}$) is called Heisenberg's uncertainty principle in quantum mechanics. The proof of Theorem 2.1 is as follows.

Proof. Note that if T is Hermitian, so is $T - cI$ where c is a real number. Also note that, $\overline{E_\psi(T)} = \overline{(\psi, T\psi)} = (T\psi, \psi) = (\psi, T\psi) = E_\psi(T)$. So the expectation value is always real. Moreover, if T is Hermitian, so is T^2. (So the standard deviation is always real too.) Moreover, if A, B are Hermitian, then $\sqrt{-1}[A, B]$ is also Hermitian. (Why?)

$$\langle \Delta A \Delta B \rangle^2 = \langle \psi, (A - E(A)I)^2 \psi \rangle \langle \psi, (B - E(B)I)^2 \psi \rangle = \langle \psi(A - E(A)I), (A - E(A)I)\psi \rangle \langle \psi(B - E(B)I), (B - E(B)I)\psi \rangle \tag{2.1}$$

$$\geq |\langle (A - E(A)I)\psi, (B - E(B)I)\psi \rangle|^2 = |\langle f, g \rangle|^2 = \left(\frac{\langle f, g \rangle + \langle g, f \rangle}{2}\right)^2 + \left(\frac{\langle f, g \rangle - \langle g, f \rangle}{2\sqrt{-1}}\right)^2 \geq \left(\frac{\langle f, g \rangle - \langle g, f \rangle}{2\sqrt{-1}}\right)^2$$

$$= \frac{1}{4}|E([A, B])|^2 \tag{2.2}$$

after simplification. □

Here is an important definition : If $v, w \in V$, they are said to be orthogonal if $\langle v, w \rangle = 0$. If S is a set of vectors, S is said to be orthogonal if any pair of vectors is orthogonal. An orthonormal set S is an orthogonal set where $\|v\| = 1$ for all $v \in S$.

Lemma 2.2. *An orthogonal set of non-zero vectors is linearly independent.*

Proof. Suppose $v_1, \ldots, v_k$ are orthogonal and $\sum_i c_i v_i = 0$. Then $\sum_i \langle c_i v_i, v_j \rangle = 0$. Thus, $c_j \|v_j\|^2 = 0$ and hence $c_j = 0 \; \forall j$. □

As a corollary of the proof (rather than the statement), if β is a linear combination of some orthogonal vectors v_i, then $\beta = \sum_k \frac{\langle \beta, v_k \rangle}{\|v_k\|^2} v_k$. Here are examples and non-examples. Also, the maximum number of orthogonal vectors is the dimension of V.

(1) The standard bases in $\mathbb{R}^n$ and $\mathbb{C}^n$ with respect to the standard inner products are orthonormal.
(2) The vector (x, y) is orthogonal to $(y, -x)$. (Rotation by 90 degrees.)
(3) The matrices $(E^{pq})_{ij} = \delta_{ip}\delta_{jq}$ form an orthonormal basis.
(4) The vectors x, x^2 are not orthogonal under the L^2 inner product. However, $\sqrt{2}\cos(2\pi nx)$, $\sqrt{2}\sin(2\pi nx)$ are orthonormal.

The following fundamental result shows that the maximum number of orthogonal vectors is actually equal to the dimension : Let V be a finite-dimensional inner product space, and let $w_1, \ldots, w_n$ be a linearly independent set. Then there exists an orthonormal basis $v_1, \ldots, v_n$ of the subspace spanned by $w_1, \ldots, w_n$. (As a corollary, evert finite-dimensional inner product space has an orthonormal basis.)

The proof is called "Gram-Schmidt orthogonalisation". Let $v_1 = \frac{w_1}{\|w_1\|}$. Assume that we have defined $v_1, \ldots, v_k$. Then define $\tilde{v}_{k+1} = w_{k+1} - \sum_{i=1}^k \langle w_{k+1}, v_i \rangle v_i$ and $v_{k+1} = \frac{\tilde{v}_{k+1}}{\|\tilde{v}_{k+1}\|}$. Note that $\langle v_{k+1}, v_j \rangle = 0$ for all $1 \leq j \leq k$. Thus, v_i are orthonormal (and hence linearly independent). Also, w_{k+1} is a linear combination of the v_i. We are done by induction.

The point of orthonormal bases is that in such a basis, $H_{ij} = \langle e_i, e_j \rangle = \delta_{ij}$ and hence $\langle v, w \rangle = v^T \bar{w}$: exactly the Euclidean expression ! What does this mean for Hermitian positive-definite matrices. For that, how does H_{ij} change if one changes the basis ? That is, $H_{new,ij} = \langle e_{new,i}, e_{new,j} \rangle =$

$\langle (P^{-1})_{ki} e_{old,k}, (\bar{P}^{-1})_{lj} e_{old,l} \rangle = P^{-1}_{ki} \bar{P}^{-1}_{lj} H_{old,kl}$, implying that $P^T H_{new} \bar{P} = H_{old}$ where $\vec{v}_{new} = P \vec{v}_{old}$. In other words, given a Hermitian positive-definite matrix H, Gram-Schmidt produces an invertible matrix P such that $H = P^T \bar{P} = Q^\dagger Q$ (where $Q = \bar{P}$). Moreover, P, Q are actually upper triangular with positive diagonal entries ! (Why ?) This decomposition is called the Cholesky decomposition. It is unique ! Indeed, if $H = Q^\dagger Q = L^\dagger L$, then $Q^\dagger = L^\dagger L Q^{-1}$, i.e., $(L^\dagger)^{-1} Q^\dagger = L Q^{-1}$, i.e., *Lower* = *Upper*. Hence, both are diagonal. Hence $L = DQ$ where D is diagonal (with positive diagonal entries). So $Q^\dagger Q = Q^\dagger D^2 Q$ and hence $D = I$ because Q has positive diagonal entries (and is hence invertible).

Note that if there is an efficient way to compute the Cholesky decomposition, then we can solve linear equations $Hx = b$ quickly. Indeed, solve $Q^\dagger y = b$ and then $Qx = y$ quickly (they are triangular matrices and hence back/forward substitution does the job).