NOTES FOR 3 OCT (THURSDAY)

1. Recap

(1) Proved the existence of tensor products (where I intentionally omitted some details) and produced examples of tensors that are not pure tensors.
(2) Proved the existence and uniqueness of a signed volume function that is multilinear, alternating, and normalised. Defined the determinant of a matrix and proved that it is multiplicative. Also proved that A is invertible iff $\det(A) \neq 0$.

2. Determinants

Now we prove the second half of the theorem.

Theorem 2.1. $\det(A^T) = \det(A)$.

Proof. If A is not invertible, neither is A^T. Hence both determinants are 0 and equal. If A is invertible, then $\det(A) = \Pi_i \det(E_i^T)$. For elementary column operations, this property is easily true. Hence it is true for A. □

As a consequence, all the properties above hold when "column" is replaced with "row". The proof of $\det(A)$ determining whether a matrix is invertible or not, actually gives us a way to calculate it. Indeed, bring A to its row echelon (or for that matter column echelon) form B. Then $B = PA$. Thus, $\det(B) = \det(P)\det(A) = (-1)^i \Pi_j c_j \det(A)$ where i is the number of row interchanges and c_j are constants that elementary row operations use to scale rows with, i.e., $R_{i_j} \to c_j R_{i_j}$. $\det(B)$ can be calculated easily. (It is 1 or 0.) Here is an example.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \tag{2.1}$$

$R_3 \to R_3 - 2R_1$, $R_3 \to R_3 + 2R_2$ gives

$$\det(A) = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{vmatrix} = 7 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 7 \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 7 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 7 \tag{2.2}$$

Now we prove a standard formula for the determinant : $\det(A) = \sum_i A_{1i}(-1)^{i+1} M_i$ (called "expanding along the first row") where M_i is the determinant of the matrix obtained by removing the 1^{st} row and i^{th} column of A. (A similar formula can be written using the other rows and columns.) Indeed, the RHS is clearly 1 for the identity matrix. Uniqueness of the determinant shows the desired formula given the following.

(1) Multilinearity : Let $C_i \to aC_i + bv_i$. Then $\det(\tilde{A}) = \sum_{k \neq i} A_{1i}(-1)^{i+1}\tilde{M}_i + (aA_{1i} + bv_i)(-1)^{1+i} M_i$. By induction hypothesis we are done.
(2) Alternation : Let $C_i = C_j = v$. Then $\det(A) = \sum_{k \neq i,j} A_{1i}(-1)^{k+1} M_k + v_i(-1)^{i+1} M_i + v_i(-1)^{j+1} M_j$. The first term is zero by induction hypothesis. The second term is zero because $M_i = -(-1)^{j-i} M_j$.

Let us return to equation 2.2 the above example. Expanding along the first column yields $\det(A) = \begin{vmatrix} 1 & 3 \\ 0 & 7 \end{vmatrix} = 7 - 0 = 7$. This leads us to an important little observation (and a definition) : An $n \times n$ matrix A is said to be upper triangular (and likewise lower triangular) if $A_{ij} = 0$ whenever $i > j$. (It is strictly upper triangular if the diagonal entries are zero as well.) The determinant of an upper triangular matrix equals the product of its diagonal entries : Indeed, expanding along the first column, $\det(A) = A_{11}M_{11}$ and M_{11} is the determinant of another upper triangular matrix. By induction on n we are done.

Using determinants we can actually find a formula for the inverse of a matrix. Define the Adjugate matrix $Adj(A)_{ij} = (-1)^{i+j}M_{ji}$. Let us compute the following.

$$(AAdj(A))_{ii} = \sum_k A_{ik}(-1)^{i+k}M_{ik} = \det(A)$$

$$\text{when } i \neq j \; (AAdj(A))_{ij} = \sum_k A_{ik}(-1)^{i+k}M_{jk} \tag{2.3}$$

To evaluate the second expression, take a new matrix N whose i^{th} column consists of the j^{th} column of A and everything else is the same as A. Then $\det(N) = 0$. Then $\sum_k Adj(N)_{ik}N_{ki} = \sum_k (-1)^{i+k} min(N)_{ki} N_{ki} = \det(N) = 0$. Note that $min(N)_{ki} = min(A)_{ki}$. Also, $N_{ki} = A_{kj}$. Hence the second expression above is 0. Thus, $AAdj(A) = \det(A)I$. The above expression gives a formula for the inverse $A^{-1} = \frac{Adj(A)}{\det(A)}$. It also allows us to derive Cramer's rule : To solve $AX = Y$ when $\det(A) \neq 0$, $X = A^{-1}Y = \frac{Adj(A)Y}{\det(A)}$. Now $(Adj(A)Y)_i = \sum_j (Adj(A))_{ij}Y_j = \sum_j (-1)^{i+j}M_{ji}Y_j$ which is simply the determinant of the matrix obtained by replacing the i^{th} column of A with Y.

Determinants seem to be a useful tool for $n \times n$ matrices. What about $m \times n$ matrices ? Even for $n \times n$ matrices, if $\det(A) = 0$, are we completely in the dark ? Actually, no. Let's take an example : $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 6 \end{bmatrix}$. The determinant of A is clearly 0. The column echelon form is easily seen to be $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 0 \end{bmatrix}$. So the rank is 2. In the original matrix, the minor $M_{11} = 1 \neq 0$. So it seems that the rank is equal to the dimension of the largest non-zero minor. In fact, this observation is correct.

Theorem 2.2. *Let $A \in Mat_{m\times n}(\mathbb{F})$. Let k be the largest integer so that there exists a $k \times k$ submatrix of A whose determinant is $\neq 0$. The rank of A equals k.*

Proof. Row and column operations do not affect the rank. We prove that they do not affect k either. Given this claim, we are done. Indeed, bringing A simultaneously to the row and column echelon form, we see the result. We prove only for elementary row operations (taking transpose clearly does not affect either quantity). Let M be a $l \times l$ minor of A and M' be the same $l \times l$ minor of A'.

(1) $R_i \Leftrightarrow R_j$: It is clear that $\det(M) = \pm\det(M')$ if either i, j are both in M or neither is in M. If only i is in M, then choose M' to have the j^{th} row of A' instead of the i^{th} row (but in the correct position). So if A has an $l \times l$ nonzero minor, then so does A' (and vice-versa). Hence k is unchanged under this operation.
(2) $R_i \to cR_i$ where $c \neq 0$: As above, $\det(M')$ is either $c\det(M)$ or $\det(M)$ (depending on whether entries from R_i are in M or not). Hence k is unchanged.

(3) $R_i \to R_i + cR_j$: If i, j are both present in M or are both absent, $det(M) = det(M')$. If only j is present, even then this is true. If only i is present, then $\det(M')$ fails to be non-zero iff the determinant obtained by replacing R_i in M' with R_j is non-zero. So take a new minor of A' where instead of R_i we have R_j. So if A has an $l \times l$ nonzero minor, then so does A' (and vice-versa because this operation is reversible). Hence k is unchanged under this operation.

□

Lastly, if $B = PAP^{-1}$, then $\det(B) = \det(A)$, i.e., given a linear transformation $T : V \to V$ where V is a finite-dimensional vector space, $\det(T)$ defined as $\det(A_{\mathcal{B}})$ where $\mathcal{B}$ is any ordered basis of V is well-defined.

3. Eigenvalues and Eigenvectors

Consider these two problems :

(1) Given a matrix R that represents a rotation in $\mathbb{R}^3$, find its axis of rotation.
(2) Solve $\frac{dx}{dt} = 2x + 3y$, $\frac{dy}{dt} = 3x + 2y$.
(3) The chance of it raining tomorrow if it rains today is 0.7 and if it does not rain, it is 0.5. Given that it rained today, what is the chance of it raining after many days ?