NOTES FOR 3 SEPT (TUESDAY)

1. Recap

(1) Gave several examples and non-examples of linear transformations.
(2) Proved that an abstract polynomial is determined by its polynomial function uniquely over fields of characteristic zero. Also proved that a polynomial over $\mathbb{R}, \mathbb{C}$ vanishes everywhere if it vanishes in a small open set.
(3) Proved nullity-rank and that row rank equals column rank (using nullity-rank).
(4) Defined direct sums and products.

2. Linear transformations

Note that the proof of nullity-rank shows that there is an isomorphism $Nullity \oplus Range \rightarrow V$. However, the isomorphism uses a specific basis. (It is as silly as saying any finite-dimensional vector space is isomorphic to $\mathbb{F}^n$.) Is there a basis-independent way of thinking about the nullity, range, and V ? The answer is through the definition of a quotient vector space. The point is that the extended basis $f_1, \ldots, f_{n-r}$ can be changed to $\tilde{f_i} = f_i + \sum_j c_{ij} e_j$ without changing the fact that it is a basis. Geometrically, consider the projection map $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\pi_1(x, y) = x$. It is a linear map. The kernel is spanned by the y-axis. To get a basis for the range, the most obvious choice (because we know how to measure angles between vectors) is $\pi_1(\hat{i})$ but we can make it $\pi_1(\hat{i} + a\hat{j}) = \pi_1(\hat{i})$. In some sense, we should consider all the vectors of the type $\hat{i} + a\hat{j}$ as the "same" in order to avoid a specific choice. We can do this neatly using an equivalence relation :
Def : Let $W \subset V$ be a subspace of V. The quotient space V/W as a set is the collection of equivalence classes under the relation $v_1 \sim v_2$ if $v_1 = v_2 + w$ where $w \in W$. This set can be given a vector space structure as $a[v_1] + b[v_2] := [av_1 + bv_2]$. Indeed this operation is well-defined (why ?) It also satisfies all the vector space axioms (why ?)
Here are examples.

(1) $W = \mathbb{R}$ is the y-axis in $V = \mathbb{R}^2$. Then V/W is the set of vectors spanned by $[\hat{i}]$ (why?) and hence $V/W \equiv \mathbb{R}$.
(2) If $W = \{\vec{0}\}$ then $V/W \equiv V$. Likewise, if $W = V$, then $V/W = \{\vec{0}\}$.
(3) If $T : V \rightarrow W$ where V, W are not necessarily finite-dimensional, then $V/ker(T) \equiv Ran(V)$. Indeed, define $[v] \rightarrow T(v)$. Why is this well-defined and an isomorphism ? (Notice that no mention of a basis is there in this definition !)
(4) If V, W are finite dimensional, then V/W is finite-dimensional and $dim(V/W) = dim(V) - dim(W)$. Indeed, let $w_1 \ldots, w_k$ be a basis of W and let $w_1, \ldots, w_k, v_1, \ldots, v_{n-k}$ be a basis for V. Then $[v_i]$ form a basis for the quotient. (Why?) This gives another proof of the nullity-rank theorem.

The collection of all linear transformations $T : V \rightarrow W$ is denoted as $L(V, W)$. This set has a rich structure on it :

(1) $L(V, W)$ is a vector space : Indeed, define the map $T + U : V \rightarrow W$ as $(T + U)(v) = T(v) + U(v)$. Clearly $T + U \in L(V, W)$. Likewise, so is $aT : V \rightarrow W$ defined as $(aT)(v) = aT(v)$. It is easy to verify that the axioms of a vector space are satisfied (with the zero map being the 0 vector).

(2) If V, W are n, m dimensional respectively, then $L(V, W)$ is finite-dimensional of dimension mn : Indeed, Let $e_1, \ldots, e_n$, $f_1, \ldots, f_m$ be bases of V and W respectively. Define the mn linear transformations $T_{ij} : V \to W$ as $T_{ij}(e_k) = \delta_{ik} f_j$. I claim that T_{ij} form a basis for $L(V, W)$. Indeed,
   (a) They are linearly independent : $\sum_{i,j} c_{ij} T_{ij} = 0$ means that $\sum_{i,j} c_{ij} T_{ij}(e_k) = 0 \ \forall\ 1 \leq k \leq n$. Thus, $\sum_j c_{kj} f_j = 0$. Since f_j are linearly independent, $c_{kj} = 0 \ \forall k, j$.
   (b) They span $L(V, W)$: Indeed, $T(v) = \sum_j v_j T(e_j) = \sum_{i,j} A_{ij} v_j f_i = \sum_{i,j} A_{ij} T_{ji}(v)$. Thus, $T = \sum_{i,j} A_{ij} T_{ji}$.

(3) Let V, W, Z be vector spaces over a field $\mathbb{F}$. If $T : V \to W$ and $U : W \to Z$ are linear maps, then $U \circ T : V \to Z$ is a linear map : Indeed, $U \circ T(cv_1 + v_2) = U(cT(v_1) + v_2) = cU \circ T(v_1) + U \circ T(v_2)$. A linear map $T : V \to V$ is sometimes called an operator on V. The notation T^n is well-defined for a composition of T with itself n times. $T^0 = I$ by definition.

Here are some examples to illustrate the $UT \neq TU$ in general.

(1) Let L_1, L_2 be the rotations (anticlockwise) by 90 degrees about the z-axis and the y-axis respectively. Then $L_2(L_1(\hat{i})) = L_2(\hat{j}) = \hat{j}$ whereas $L_1(L_2(\hat{i})) = L_1(-\hat{k}) = -\hat{k} \neq \hat{j}$.

(2) Let V be the vector space of polynomials functions $\mathbb{R} \to \mathbb{R}$, $D : V \to V$ be the derivative and $M_x : V \to V$ be multiplication by x. Then $(L_x D - DL_x)(p(x)) = xp' - (xp)' = -p$, i.e., $DL_x - L_x D = Id$.