NOTES FOR 5 SEPT (THURSDAY)

1. Recap

(1) Defined V/W where $W \subset V$ and proved that if V, W are f.d., then so is V/W and its dimension is $dim(V) - dim(W)$.
(2) Proved that $V/ker(T) \equiv Range(T)$.
(3) Proved that $L(V, W)$ is a vector space and is f.d. if V, W are, with dimension $dim(V)dim(W)$. Proved that $U \circ T$ is a linear transformation and that $UT \neq TU$ in general (with counterexamples).

2. Linear transformations

Recall that an invertible linear transformation $T : V \to W$ is an isomorphism, i.e., its inverse is also linear. (Similar to matrices, one has the notions of left and right inverses as well.) Moreover, if $A : V \to W$ and $B : U \to V$ are invertible, then $AB : U \to W$ is also invertible with inverse $B^{-1}A^{-1}$.

Def : A linear map $T : V \to W$ is called non-singular if $ker(T) = \{\vec{0}\}$. Note that T is $1-1$ iff it is non-singular : Indeed, $T(v_1) = T(v_2) \Leftrightarrow T(v_1 - v_2) = 0$ and $v_1 = v_2$ iff $ker(T) = \{0\}$.

Proposition 2.1. *Let $T : V \to W$ be linear. Then T is non-singular iff it carries linearly independent subsets to linearly independent subsets.*

Proof. If it is non-singular : Suppose $\sum_i c_i T(v_i) = 0$ then $T(\sum_i c_i v_i) = 0$ and hence $\sum_i c_i v_i = 0$ which means (by linear independence of v_i) that $c_i = 0\ \forall i$.
If it carries lin. indep. to lin. indep. : If $T(v) = 0$ where $v \neq 0$, then $\{v\}$ goes to something not lin. indep. A contradiction. ☐

Consider these examples.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(v, w) = (v+w, v)$. Then $T(v, w) = (0, 0)$ implies that $(v, w) = (0, 0)$. So T is non-singular. Moreover, T is onto because $T(y, x - y) = (x, y)$. In fact, $T^{-1}(a, b) = (b, a - b)$.
(2) Let V be the space of polynomial functions from $\mathbb{R}$ to itself. Then $D : V \to V$ (the differentiation map) is linear, and $Dp(x) = 0$ iff $p(x) = c$. So D is singular. However, consdier $E(p(x)) = p_0 x + \frac{1}{2} p_1 x^2 + \ldots$. This map is linear and $DE = Id$. However, $ED(1) = E(0) = 0 \neq 1$. So it is right invertible but not left invertible.
(3) $M_x : V \to V$ is non-singular. Consider $U(p(x)) = p_1 + p_2 x + \ldots$. Then $UM_x(p(x)) = U(p_0 x + p_1 x^2 + \ldots) = p_0 + p_1 x + \ldots$. However, $M_x U(1) = M_x(0) = 0$. So it is left but not right invertible and is non-singular.
(4) $T : \mathbb{R} \to \mathbb{R}^2$ defined by $T(x) = (x, 0)$ is certainly non-singular but not onto.

So it appears that for infinite-dimensional spaces, non-singular has not much to do with invertibility. However, it appears to be true for finite-dimensions (with the same dimension). (Akin to saying that if $f : X \to Y$ is $1-1$ and X, Y are finite sets of the same size, then f is onto.) Indeed,

Theorem 2.2. *Let V, W be finite-dimensional vector spaces with the* same *dimension. If $T : V \to W$ is a linear map, then TFAE.*

(1) *T is invertible.*

(2) *T is non-singular.*
(3) *T is onto.*

Proof. Let $n = dim(V) = dim(W)$ and $e_1, \ldots, e_n$ be a basis of V. Of course $1 \Rightarrow 2, 3$. We prove that $2 \Rightarrow 3, 1$: Indeed, if T is non-singular (it is surely $1-1$), then $T(e_1), \ldots, T(e_n)$ is a basis of W. Hence, $Rank(T) = n = dim(W)$ and hence T is onto. (Therefore, T is invertible.)

Now $3 \Rightarrow 2$: $nullity(T) = 0$ and hence T is non-singular. □

Now we study the relationship between linear maps and matrices.

Theorem 2.3. *Let V, W be n, m-dimensional vector spaces respectively over a field $\mathbb{F}$. Let $\mathcal{B}, \mathcal{B}'$ be ordered bases for V, W respectively. For each $T \in L(V, W)$ there exists an $m \times n$ matrix A such that $\vec{Tv}_{\mathcal{B}'} = A\vec{v}_{\mathcal{B}}$ for all $v \in V$. (Such an A is called the matrix of T in the bases $\mathcal{B}, \mathcal{B}'$).*
Moreover, $T \to A$ is an isomorphism between $L(V, W)$ and $Mat_{m\times n}(\mathbb{F})$.

Proof. $T(v) = T(\sum_j v_j e_j) = \sum_j v_j T(e_j) = \sum_{i,j} v_j A_{ij} f_i$ for some unique collection of coefficients A_{ij}. Hence $\vec{Tv}_{\mathcal{B}'} = A\vec{v}_{\mathcal{B}}$. Clearly, $T \to A$ is linear. Moreover, given any matrix A, $T(v) := \sum_{i,j} v_j A_{ij} f_i$ is a linear map. Thus $T \to A$ is an isomorphism. □

When $V = W$, very often, $\mathcal{B}$ is taken to equal to $\mathcal{B}'$ in which case the case, the matrix A corresponding to T is written as $[T]_{\mathcal{B}}$. Here are some examples.

(1) Let $T : V = \mathbb{F}^n \to W = \mathbb{F}^m$ be $T(v) = Av$. Then, if V, W are equipped with their standard bases, the matrix corresponding to T is A itself.
(2) Let V be the space of all polynomial functions of degree ≤ 3, $f_p(x) : \mathbb{R} \to \mathbb{R}$. Then define $D : V \to V$ by differentiation. Then $D(1) = 0$, $D(x) = 1$, $D(x^2) = 2.x$, $D(x^3) = 3.x^2$. Hence, $[D] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(3) In the previous example, choose the domain basis to be standard and the range basis to be $e'_1 = 1, e'_2 = x, e'_3 = x^2 - 1, e_4 = x^3 - 3x$. Then $D(1) = 0, D(x) = e'_1, D(x^2) = 2e'_2, D(x^3) = 3x^2 = 3e'_3 + 3e'_1$. So $[D] = \begin{pmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Here is an important result about composition of linear maps.

Proposition 2.4. *Let V, W, Z be finite-dimensional vector spaces over a field $\mathbb{F}$. Let $T : V \to W$, $U : W \to Z$ be linear maps. Let $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ be ordered bases for these three spaces. Let A, B be the matrices of T, U with respect to these bases. Then the matrix of $U \circ T$ with respect to $\mathcal{B}, \mathcal{B}'$ is $C = BA$.*

Proof. Indeed, $\vec{Tv}_{\mathcal{B}'} = A\vec{v}_{\mathcal{B}}$ and $\vec{Uw}_{\mathcal{B}''} = B\vec{w}_{\mathcal{B}'}$. Thus, $U \vec{\circ} Tv_{\mathcal{B}''} = B\vec{Tv}_{\mathcal{B}'} = BA\vec{v}_{\mathcal{B}}$. □

Note that if $V = W$ and the basis is chosen to be the same on both sides, then an operator $T : V \to V$ gives a unique $n \times n$ matrix A (this correspondence is an isomorphism). Moreover, products are preserved. As a consequence, if T is invertible, i.e., $UT = TU = I$, then so is A (and vice-versa), and T^{-1} is represented by A^{-1}. Now we see what happens when we change the bases involved.

Theorem 2.5. *Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Let $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}, \mathcal{B}' = \{e'_1, \ldots\}$ be two ordered bases. Let $T : V \to V$ be an operator. If $P = [P_1 P_2 \ldots P_n]$ is a $n \times n$ matrix with* *columns* $P_i = \vec{e}'_{i\mathcal{B}}$ *then* $[T]_{\mathcal{B}} = P[T]_{\mathcal{B}'}P^{-1}$. *Alternatively, if* U *is the operator defined by* $Ue_i = e'_i$, *then* $[T]_{\mathcal{B}'} = [U]_{\mathcal{B}}[T]_{\mathcal{B}}[U]^{-1}_{\mathcal{B}}$.

Proof. The i^{th} column A_i of $[T]_{\mathcal{B}}$ is $T\vec{e}_{i\mathcal{B}}$ which is equal to $PT\vec{e}_{i\mathcal{B}'} = P[T]_{\mathcal{B}'}\vec{e}_{i\mathcal{B}'} = P[T]_{\mathcal{B}'}P^{-1}\vec{e}_{i\mathcal{B}}$ which is the i^{th} column of $P[T]_{\mathcal{B}'}P^{-1}$. Clearly this statement is the same as the alternate statement using U. □

This theorem motivates the following definition : Let A, B be two $n \times n$ matrices. Then A is said to be similar to B if there exists an invertible $n \times n$ matrix P such that $A = PBP^{-1}$. (Note that similarity is an equivalence relation.)

It is not hard to see that $A \sim B$ iff the linear transformation $v \to Bv$ from $\mathbb{F}^n$ to itself in the basis obtained from the columns of P^{-1} is expressed by the matrix A.

Here are some examples.

(1) Consider $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $\pi_1(x, y) = x$. In the standard basis, the matrix for this transformation is $[A] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Suppose we take a new basis $e_1 = \hat{i} + \hat{j}, e_2 = 2\hat{i} + \hat{j}$ (why is this a new basis ?), then let P^{-1} be the matrix whose columns are $[e_1 e_2] = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Then $P = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ (why is this P?) Hence in the new basis, $[B] = PAP^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$. Indeed, $\pi_1(e_1) = \hat{i} = e_2 - e_1$ and $\pi_1(e_2) = 2\hat{i} = 2(e_2 - e_1)$.

(2) Let V be the space of polynomial functions $\mathbb{R}$ to itself with degree ≤ 3. Let $D : V \to V$ be the derivative operator. Let $\mathcal{B}' = \{1, x, x^2, x^3\}$ and $\mathcal{B} = \{1, x + t, (x + t)^2, (x + t)^3\}$. Note that $e_1 = e'_1, e_2 = e'_2 + te_1, e_3 = e'_3 + 2te'_2 + t^2 e'_1, e_4 = e'_4 + 3te'_3 + 3t^2 e'_2 + t^3 e'_1$. Let P^{-1} be the matrix whose columns are $[e_1 e_2 e_3 e_4]$. We know D in the basis $\mathcal{B}'$ and hence in the basis $\mathcal{B}$, it is PDP^{-1}. However, a simpler way is to use the definition : $De_1 = 0, De_2 = 1 = e_1, De_3 = 2e_2, De_4 = 3e_3$. Interestingly, this illustrates that PDP^{-1} can be equal to D.