NOTES FOR 7 NOV (THURSDAY)

1. Recap

(1) Studied orthogonal projections.
(2) Finite-dimensional little Riesz representation theorem and adjoints.

2. Linear functionals and adjoints

An operator (between finite-dimensional vector spaces) is called self-adjoint or Hermitian if $T = T^*$, i.e., $\langle v, Tw\rangle = \langle Tv, w\rangle$. This definition coincides with our earlier definition. Please note that in infinited dimensions, self-adjoint is far more subtle (basically because the domains need to be considered carefully).

Note that when V is a finite-dimensional vector space, $T \to T^*$ is an anti-linear isomorphism between $L(V,V)$ and itself. Indeed, $\langle v, (aU + bT)^*(w)\rangle = \langle (aU + bT)v, w\rangle = a\langle v, U^*w\rangle + b\rangle v, T^*w\rangle = \langle v, (\bar{a}U^* + \bar{b}T^*)w\rangle$. Moreover, it is easy to see that $(UT)^* = T^*U^*$ and $(T^*)^* = T$. Also note that every linear operator T can be written as $\frac{T+T^*}{2} + \sqrt{-1}\frac{T-T^*}{2\sqrt{-1}}$. (Just like $z = x + \sqrt{-1}y$.)

To look at the matrix formulation of the above, firstly, if e_i is an ordered orthonormal basis of V, then the matrix of T in that basis is $A_{ji} = \langle Te_i, e_j\rangle$. Indeed, $Te_i = A_{ji}e_j$. Hence, $\langle Te_i, e_j\rangle = A_{ji}$. So $(A^*)_{ji} = \langle T^*e_i, e_j\rangle = \langle e_i, Te_j\rangle = \overline{A_{ij}}$. Hence, $A^* = A^\dagger$. So a Hermitian operator is represented by a Hermitian matrix *in an orthonormal basis*. Here are examples of adjoints.

(1) Consider the linear map $L_M : Mat \to Mat$ given by $L_M(A) = MA$. Then, equipped with the usual inner product $\langle A, B\rangle = tr(AB^\dagger)$, its adjoint is $tr(L_M(A)B^\dagger) == tr(MAB^\dagger) = tr(AB^\dagger M)tr(A(L_M^*B)^\dagger)$. Hence $L_M^*B = M^\dagger B$.
(2) Let V be the space of polynomial functions on $[-1,1]$ of degree ≤ 10 equipped with the L^2 inner product. Let $T : V \to V$ be the operator $T(p) = \sqrt{-1}p'$. Then $(Tp, g) = \int_{-1}^{1} \sqrt{-1}p'\bar{g}dx = \sqrt{-1}p(1)\bar{g}(1) - \sqrt{-1}p(0)\bar{g}(0) - \int_{-1}^{1} \sqrt{-1}p\bar{g}'dx = \sqrt{-1}p(1)\bar{g}(1) - \sqrt{-1}p(-1)\bar{g}(-1) + (p, Tg)$. So it is not self-adjoint unless the boundary term is 0 (for instance if we restrict ourselves to polynomials that vanish at the end-points).

3. Norms of operators

We digress a bit to discuss how much operators "stretch" vectors. Before that, define two norms to be equivalent if there exist positive constants m, M such that $m\|v\|_1 \leq \|v\|_2 \leq M\|v\|_1 \ \forall\ v \in V$. Here is a little observation.

Lemma 3.1. *On finite-dimensional vector spaces, any two norms are equivalent.*

Proof. We shall prove that every norm is equivalent to one that is induced by a fixed inner product (whose norm is $\|.\|_0$). Indeed, choose some inner product (by isomorphism with $\mathbb{C}^n$ for instance) and take an orthonormal basis $e_1, \ldots, e_n$. The unit sphere S_0 in this norm is compact. (Indeed, it is the unit sphere in $\mathbb{C}^n$ in the orthonormal basis.) Moreover, $\|.\|_1$ is a continuous function in the $\|.\|_0$ topology. Indeed, $\|x_n - x\|_1 \leq |(x_n - x)_i|\|e_i\|_1 \leq C|x_n - x|_i \leq C\sqrt{n}\|x_n - x\|_0$. Hence, the function $\|.\|_1$

achieves a minimum m on S_0 (which is not zero). Thus, $\|v\|_1 = \|v\|_0\|\frac{v}{\|v\|_0}\|_1 \geq m\|v\|_0$. Likewise for the maximum. □

As a corollary, we see that the unit ball is compact for every finite-dimensional normed vector space. (In fact, if reasonable restrictions are imposed on the norm, this property is true strictly for finite-dimensional spaces.)
If $T : V \to W$ is a linear map between two normed vector spaces, then the operator norm of T is defined as $\|T\| = \sup_{\|v\|=1} \|Tv\|$. If V and W are finite-dimensional, then the supremum is a maximum by compactness of the unit ball. Hence, on the space $Mat_{n\times n}(\mathbb{C})$, there are several possible (equivalent) equivalent norms : The operator norm, The norm from the inner product on $\mathbb{C}^{n^2}$, and the taxi-cab norm $\|A\| = \sum_{i,j} |A_{ij}|$. However, the equivalence might change with n. For instance, the operator norm of I is 1, whereas in the other norms it is $\sqrt{n}, n$, which grow with increasing n. Note that if A is invertible, then $1 \leq \|A\|_{op}\|A^{-1}\|_{op}$.

Here is an interesting and useful lemma.

Lemma 3.2. *Let $A \in Mat_{n\times n}(\mathbb{C})$.*

(1) *If $\|A\|_{op} < 1$, then $I - A$ is invertible.*
(2) *The series $e^A = \sum_k \frac{A^k}{k!}$ converges. Moreover, $e^{A+B} = e^A e^B$ if A and B commute. Also, the function e^{At} is differentiable for all $t \in \mathbb{R}$ with the derivative being Ae^{At}.*

Proof. (1) Let $B = \sum_k A^k$. This series converges in the operator norm topology. Indeed, $\|\sum_{k=N}^{M} A^k\|_{op} \leq \sum_{k=N}^{M} \|A\|^k \leq \frac{\|A\|^N}{1-\|A\|} \to 0$ as $N \to \infty$. So at least B is a Cauchy sequence. Since all norms over f.d. spaces are equivalent, this space is complete and B converges (in fact, in all other norms too). So, in particular, the elements of B converge absolutely. Now, $(I-A)B = B - AB = \sum_k A^k - \sum_k A^{k+1}$. By absolute convergence, any order of summation gives the same result. Hence $(I - A)B = I$.

(2) Akin to the above, $\|\sum_{k=N}^{M} \frac{A^k}{k!}\| \leq \sum_{k=N}^{M} \frac{\|A\|^k}{k!}$ which is of course Cauchy and hence converges (and hence does so absolutely, entrywise). The property $e^{A+B} = e^A e^B$ if A and B commute has the same proof as for numbers. Now, $\lim_{h\to 0} \frac{e^{A(t+h)}-e^{At}}{h} - Ae^{At} = e^{At}(\lim_{h\to 0} \frac{e^{Ah}-I-hA}{h}$. Now, $\|\frac{e^{Ah}-I-hA}{h}\| = \|h(\frac{A}{2!} + \ldots)\| \leq |h|(\frac{\|A\|}{2!} + \ldots) \to 0$ as $h \to 0$.

□

As a corollary,

Proposition 3.3. *The differential equation $\frac{dv}{dt} = Av$ with $v(0) = v_0$ has a unique solution $v = e^{At}v_0$.*

Proof. Note that $\frac{d(e^{-At}v)}{dt} = 0$. Hence, $v = e^{At}v_0$. □

4. Unitary operators; Spectral theorem for self-adjoint operators

Let V, W be inner product spaces. A linear map $T : V \to W$ is said to preserve inner products if $(Tu, Tv) = (u, v)$ for all $u, v \in V$. If T is an isomorphism, it is said to be a unitary isomorphism. If V and W are the same, it is called a unitary operator.

Note that a map is inner-product preserving iff it takes an orthornormal basis to an orthonormal set. Indeed, $(e_i, e_j) = (Te_i, Te_j) = \delta_{ij}$ if e_i is an orthonormal basis and T is inner-product preserving. Conversely, $(Tv, Tw) = (\sum_i v_i T(e_i), \sum_j w_j T(e_j)) = v_i \bar{w}_j = (v, w)$ if it takes an orthonormal basis to an orthonormal set. Hence, if V is finite-dimensional, then an inner-product preserving operator is unitary. The product of two unitaries is a unitary. Moreover, clearly inner-product preserving maps preserve the norms too. In fact, a linear norm-preserving map is inner product preserving (linearity is crucial). Indeed, this follows from a polarisation identity.

Another point is :

Proposition 4.1. *Let U be a unitary operator on an inner product space V. Then U has an adjoint, i.e., an operator $U^* : V \to V$ such that $(Ux, y) = (x, U^* y)\ \forall\ x, y \in V$ and it satisfies $UU^* = U^* U = I$.*

Proof. U has an inverse U^{-1}. Now $\|U^{-1}x\| = \|U(U^{-1})x\| = \|x\|$ and hence U^{-1} is unitary too. Now $(x, U^{-1}y) = (U^{-1}Ux, U^{-1}y) = (Ux, y)$. Thus, $U^* = U^{-1}$. □

If e_i is an orthonormal basis and A is the matrix of U in that basis, then $(x, y) = (Ux, Uy) = (Ax)^T \overline{Ay}$. Thus, $A^\dagger A = I$ and likewise, $AA^\dagger = I$. Such a matrix is called a unitary matrix. A real unitary matrix is called an orthogonal matrix.