HW 2

1. A commutative ring $(R, +, \times, 0, 1)$ is a set with functions $\times : R \times R \to R$, $+ : R \times R \to R$, and elements $0, 1 \in R$ such that $(R, +, 0)$ is an Abelian group (i.e., commutativity, distributivity, 0 is the identity, and inverses exist), $(R, \times, 1)$ satisfies commutativity, distributivity, and that 1 is the multiplicative identity. (The only difference from a field is that inverses do not necessarily exist.)

   (a) Define the notion of a polynomial ring $R[x]$.

   (b) Inductively define $\mathbb{F}[x_1, \ldots, x_n]$ where $\mathbb{F}$ is a field.

   (c) Prove that $\mathbb{F}[x_1, \ldots, x_n]$ is a vector space.

   (d) Define the map $p(x_1, \ldots, x_n) \to p(x_1 + a_1, \ldots, x_n + a_n)$ (where $a_i \in \mathbb{F}$) appropriately and prove that it is a linear isomorphism.

   (e) Show that an abstract polynomial with integer coefficients can be interpreted as an abstract polynomial with coefficients in a given commutative ring R.

   (f) Prove that if a polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ with integer coefficients is zero, then the corresponding polynomial function $R^n \to R$ is identically zero.

   (g) (Optional) Prove that a polynomial function over an infinite field uniquely determines the corresponding abstract polynomial.

2. Prove that interchanging rows of *any* matrix (not just column echelon) does not change the column rank.

3. Let $\mathbb{F}$ be a finite field with characteristic p where p is a prime, i.e., p is the smallest number such that $1 + 1 + \ldots + 1 (ptimes) = 0$. Prove that $\mathbb{F}$ is a vector space over the field $\mathbb{Z}_p$. Conclude that there is an n such that the size of $\mathbb{F}$ is p^n.

4. How many proper subspaces are there in the vector space $\mathbb{Z}_p^n$?

5. How many invertible $n \times n$ matrices whose entries are in $\mathbb{Z}_p$ are there ?

6. (Hoffman and Kunze) Let W_1, W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V, there are unique vectors $v_1 \in W_1, v_2 \in W_2$ such that $\alpha = v_1 + v_2$. (In this situation, V is said to be a (n internal) direct sum of W_1, W_2 and written as $V = W_1 \oplus W_2$.) Generalise this phenomenon to k-subspaces $W_1, W_2, \ldots, W_k$.