HW 3

1. (Hoffman and Kunze) Desrcibe explicitly a linear transformation from $\mathbb{R}^3$ to itself which has as its range the subspace spanned by $(1, 0, -1)$ and $(1, 2, 2)$.

2. (Hoffman and Kunze) Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $rank(T^2) = rank(T)$. Prove that the range and the kernel of T have only the zero vector in common.

3. (Hoffman and Kunze) Let $\theta \in \mathbb{R}$. Prove that the following two matrices are similar over $\mathbb{C}$:
$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ and $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$.

4. Once upon a time, in the city state of Odd Town, there lived n residents. They formed clubs, subject to the following rules:

   (a) Each club consists of an odd number of residents.

   (b) The number of members common to two different clubs is always even.

   Note that the second condition allows two clubs to be disjoint (after all, zero is an even number). Note also that the two conditions together imply that two different clubs cannot have the same set of members. The question is: what is the maximum number of clubs that they could form? (Hint : Note that a club can be thought of as a vector in $\mathbb{Z}_2^n$ by putting 0 when a member does not belong and 1 otherwise. Are the club vectors linearly independent ?)

5. A number $\alpha \in \mathbb{C}$ satisfying an equation of the form

   $$a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n = 0$$

   where $a_i \in \mathbb{Q}$ is called an algebraic number. (For instance, $\sqrt{2}$ is algebraic. It can be proven (not easy!) that e is not algebraic.)
   Let α be an algebraic number. Define the set $\mathbb{Q}[\alpha]$ to consists of all finite linear combinations of complex numbers of the form $q\alpha^m$ where $q \in \mathbb{Q}$. (For instance, $1 + \alpha, 2.5 + 0.33\alpha + 8.5\alpha^2$, etc are all in $\mathbb{Q}[\alpha]$.)

   (a) Prove that $\mathbb{Q}[\alpha]$ is a field.

   (b) Prove that $\mathbb{Q}[\alpha]$ is also a finite-dimensional vector space over $\mathbb{Q}$. What is its dimension ?

(c) Let β be another algebraic number. Denote by $(\mathbb{Q}[\alpha])[\beta]$ the set consisting of all finite linear combinations of complex numbers of the form $x\beta^m$ where $x \in \mathbb{Q}[\alpha]$. (So for instance, $1 + (1 + \alpha)\beta + \alpha^2\beta^2$ is in this set.) Prove that $(\mathbb{Q}[\alpha])[\beta]$ is a finite-dimensional vector space over $\mathbb{Q}[\alpha]$.

(d) Prove that $(\mathbb{Q}[\alpha])[\beta]$ is also a finite-dimensional vector space over $\mathbb{Q}$.

(e) Prove that $\alpha + \beta$ is algebraic.