HW 5

1. (a) Prove that $p_A(\lambda)$ is a universal polynomial in λ *and* the entries of the $n \times n$ matrix A with integral coefficients, i.e., not only is it a polynomial, but also the coefficients do not depend on anything other than n.

   (b) Assume the Cayley-Hamilton theorem holds for all complex $n \times n$ matrices A, i.e., for every $A \in Mat_{n\times n}(\mathbb{C})$, $p_A(A) = 0$. Now prove that it holds even for matrices with entries in a commutative ring.

   (c) (Optional) Prove that there exists an open set $U \subset Mat_{n\times n}(\mathbb{C}) = \mathbb{C}^{n^2}$ such that every element of U is diagonalisable. Now prove that the Cayley-Hamilton theorem holds for all complex matrices.

2. (Hoffman and Kunze) Prove that if $A, B \in Mat_{n\times n}(\mathbb{F})$, then AB and BA have the same eigenvalues in $\mathbb{F}$.

3. (Hoffman and Kunze) Let A be a 2×2 complex matrix. Prove that A is similar to either a diagonal matrix or an upper triangular matrix with 1 as the off-diagonal entry.

4. (Hoffman and Kunze) Let A be a diagonal $n \times n$ matrix. Let V be the space of $n \times n$ matrices B such that $AB = BA$. Prove that $dim(V) = a_1^2 + a_2^2 + \ldots a_k^2$ where a_i is the algebraic multiplicity of the i^{th} eigenvalue of A.

5. Let $A_{ij} = a_i b_j$ where $a_i, b_j \in \mathbb{F}$. Determine the eigenvalues and eigenvectors of this matrix. Using this data, calculate $\det(A)$.