HW 6

1. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{bmatrix}$. Find matrices P and J such that $P^{-1}AP = J$ where J is the Jordan Canonical Form of A.

2. Solve the system of ODE given by $\frac{dv}{dt} = A\vec{v}$ where A is as in problem 1.

3. Prove that if A_i is a family of commuting operators from a finite-dimensional complex vector space V to itself, then there exists a basis of V such that all the A_i are simultaneously upper-triangular.

4. (Hoffman and Kunze) Let V be the space of $n \times n$ matrices over a field $\mathbb{F}$. Let $A \in V$ and $T, U : V \to V$ be operators defined by $T(B) = AB$ and $U(B) = AB - BA$. If A is diagonalisable over $\mathbb{F}$, then are the following true or false ?

   (a) T is diagonalisable.

   (b) U is diagonalisable.

5. Prove the following.

   (a) If $a, b \geq 0$ and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. (Hint : ab is unchanged if replace $a \to ta, b \to \frac{b}{t}$ whereas the right hand side changes. Minimise the right hand side over all positive t.)

   (b) Define the map $\|.\|_p : \mathbb{R}^n \to \mathbb{R}$ as $\|x\|_p = (|x_1|^p + |x_2|^p + \ldots)^{1/p}$. Prove that $|\sum_i u_i v_i| \leq \|u\|_p \|v\|_q$ where p, q are as above and $u, v \in \mathbb{R}^n$.

   (c) Let $p \geq 1$. Prove that $\|.\|_p$ is a norm on $\mathbb{R}^n$. Prove that it is induced from an inner product iff $p = 2$.

6. (Optional) Suppose

   (a) J is an $n \times n$ real matrix such that $J^2 = -I$. Prove that n is even.

   (b) A, B are $n \times n$ real matrices such that $AB - BA$ is invertible and $A^2 + B^2 = AB$. Prove that n is a multiple of 3.