HW 7

1. Prove that the polarisation identity for normed complex vector spaces implies that the norm comes from an inner product.

2. Let V be a finite-dimensional inner product space and let $T_i : V \to V$ be a family of commuting self-adjoint operators. Prove that there exists an orthonormal basis of T such that all T_i are simultaneously diagonal in this basis.

3. Let A be an $m \times n$ complex matrix. What is the rank of $A^\dagger A$ in terms of the rank of A? Also, prove that there are unitary matrices U, V such that $U^\dagger A A^\dagger U$ and $V^\dagger A^\dagger A V$ are diagonal, and $U^\dagger A V = \Sigma$ is a "diagonal matrix" (whatever that means in this context) consisting of non-negative numbers (called the singular values of A).

4. (Artin) Let $z = e^{2\pi i/n}$ and let A be the $n \times n$ matrix whose entries are $a_{jk} = z^{jk}/\sqrt{n}$. Prove that A is unitary.

5. For every $n \times n$ complex matrix A, prove that there are four unitary matrices U_i (which may depend on A) and four complex numbers c_i (which may again depend on A) such that $A = \sum_{i=1}^{4} c_i U_i$.