NOTES FOR 1 OCT (TUESDAY)

1. Recap

(1) Defined $V \otimes W$ through a universal property and proved uniqueness up to isomorphism.
(2) Calculated the dimension of $V \otimes W$. Started proving existence as the quotient $F(V \times W)/Z$.
(3) A small point : Our earlier definition of alternating multilinear is problematic when $\mathbb{F}$ has characteristic 2. The correct way to rectify this point is by defining alternating as $T(v_1, v_2, \ldots) = 0$ if $v_i = v_j$ for some $i \neq j$. This definition implies the previous one but is equivalent to it only when $char(\mathbb{F}) \neq 2$.

2. Tensor products

Proof. (Continued..) Now if $L : V \times W \to Z$ is a bilinear map, then define $\tilde{L} : V \otimes W \to Z$ on pure tensors first as $\tilde{L}(\pi(v, w)) = L(v, w)$, i.e., $\tilde{L}(v \otimes w) = L(v, w)$. This definition makes sense because if $[(v, w)] = [(v', w')]$, then $(v, w) = (v', w')+$ a linear combination of the relations. However, $L(relations) = 0$ by bilinearity. Since every tensor is a finite linear combination of pure tensors, we can extend $\tilde{L}$ linearly in a unique manner to all of $V \otimes W$. Uniqueness follows from the construction. □

Here is an example : $\mathbb{R} \otimes \mathbb{R} \equiv \mathbb{R}$. Indeed, their dimensions match up. More concretely, if V has an ordered basis e_i and W has an ordered basis f_j, then consider the set $e_i \otimes f_j$. We claim this set is a basis of $V \otimes W$. Indeed, firstly it is linearly independent : If $\sum_{i,j} c_{i,j} e_i \otimes f_j = 0$ (a finite linear combination), then consider the bilinear map $L_{a,b}(v, w) = v_a w_b$. It factors uniquely through a linear map that satisfies $0 = \tilde{L}_{a,b}(\sum_{i,j} c_{i,j} e_i \otimes f_j) = \sum_{i,j} c_{i,j} \tilde{L}_{a,b}(e_i \otimes f_j) = \sum_{i,j} c_{i,j} \delta_{ia} \delta_{jb} = c_{a,b}$. Hence all the $c_{i,j} = 0$. Secondly, every vector in $V \otimes W$ is a finite linear combination of vectors of the form $v \otimes w = \sum_{i,j} v_i w_j e_i \otimes f_j$.

3. Determinants

Given $ax + by = c$, $dx + ey = f$, it is easy to see that the formula for x, y involves $ad - bc$ as the denominator (implying that if $ad - bc = 0$, we are in trouble). A geometric viewpoint is as follows $(x, y) \to A(x, y)$ is a linear transformation. If this linear transformation crushes a square to a line, there is no way it is surjective (and even injectivity fails). The signed area of the image of a square is indeed $ad - bc$. Likewise, one can expect that the volume of a figure formed by the columns of an $n \times n$ matrix A to play a role in the invertibility or the lack of thereof. It seems reasonable to expect that the signed volume will change sign if the columns are interchanged. It will also be $= 1$ for the standard basis. Shockingly enough, these properties are almost enough to determine a formula for the signed volume !

Theorem 3.1. *Let V be a finite-dimensional vector space of dimension n, and let $\mathcal{B}$ be an ordered basis. There exists a unique alternating multilinear map* $\det : V \times V \times V \ldots$ (n *times*) $\to \mathbb{F}$ *such that* $\det(e_1, \ldots, e_n) = 1$.

Proof. Since alternating implies (and is implied by when $char(\mathbb{F}) \neq 2$) the sign property, if such a det function existed, then it ought to satisfy

$$\det(v_1, v_2, \ldots, v_n) = \det(\sum_{i_1} (v_1)_{i_1} e_{i_1}, \sum_{i_2} (v_2)_{i_2} e_{i_2}, \ldots) = \sum_{i_1, \ldots} v_{i_1} v_{i_2} \ldots \epsilon_{i_1 i_2 \ldots}$$

where $\epsilon_{i_1 i_2 \ldots} = \det(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) = (-1)^{sgn(\sigma)}$ when all the i_j are distinct and 0 when two of them coincide.
Actually, we can use this necessary condition as the unique definition of the determinant map. Indeed, the normalisation condition is easily seen to be true. So is multilinearity. As for alternation, $\det(v, \ldots, v, \ldots) = \sum_{i_1 \ldots} v_{i_1} (v_2)_{i_2} \ldots v_{i_j} \ldots \epsilon_{i_1 i_2 \ldots} = -\sum v_{i_j} \ldots \epsilon_{i_j i_2 \ldots i_1 \ldots}$. We can assume without loss of generality that all the i_j are distinct. For every term where $i_1 < i_j$, there exists another copy with i_1, i_j interchanged. Hence $\det(v, \ldots, v, \ldots) = 0$. □

As a corollary, we can see if that ordering of the basis changes, then $\det(e_{\sigma(1)}, \ldots) = (-1)^{sgn(\sigma)}$. When A is an $n \times n$ matrix, we define $\det(A)$ to be $\sum_{i_1 \ldots} \det(A_{i_1 1} e_{i_1}, A_{i_2 2} e_{i_2}, \ldots)$. This determinant function on matrices satisfies the following properties.

Theorem 3.2. (1) $\det(AB) = \det(A)\det(B)$.
(2) *If A' is obtained from A by $C_i \to C_i + kC_j$ then* $\det(A') = \det(A)$.
(3) *If columns are interchanged, we pick up a sign in the determinant.*
(4) *If $C_i \to kC_i$, then the determinant scales by k.*
(5) *If $C_i = kC_j$ then* $\det = 0$.

Proof. The last four properties are trivially true by the above theorem. Only the first is non-trivial. Indeed, for the first property,

$$\det(AB) = \sum_{i_1 \ldots, k_1 \ldots} \det(A_{i_1 k_1} B_{k_1 1} e_{i_1}, \ldots) = \sum_{i_1 \ldots, k_1 \ldots} A_{i_1 k_1} A_{i_2 k_2} B_{k_1 1} \ldots \epsilon_{i_1 i_2 \ldots}$$

Now $\det(\sum_{i_1} A_{i_1 k_1} e_{i_1}, \ldots) = (-1)^{sgn(k \to 1)} \det(\sum_{i_1} A_{i_1 1} e_{i_1}, \ldots) = (-1)^{sgn(k)} \det(A)$ (interchanging columns picks a sign).

$$\det(AB) = \det(A) \sum_K (-1)^{sgn(k)} B_{k_1 1} \ldots = \det(A)\det(B). \tag{3.1}$$

□

The above theorem also tells us the determinants of elementary column operations. Moreover,

Theorem 3.3. *A is invertible iff* $\det(A) \neq 0$. *Moreover, if A is invertible,* $\det(A^{-1}) = 1/\det(A)$.

Proof. If A is invertible, then $A = E_k^T \ldots E_1^T$ where E_i^T are elementary column operations. Since $\det(E_i^T) \neq 0$, we see that $\det(A) \neq 0$. If A is not invertible, then the row-echelon form of A has at least one column of zeroes. Hence $\det(A) = 0$.

□