MA 229/MA 235 - Lecture 18

IISc

Recap

Recap

- Defined integral curves and proved existence/uniqueness.

Recap

- Defined integral curves and proved existence/uniqueness.
- Proved that compactly supported vector fields are complete.

Recap

- Defined integral curves and proved existence/uniqueness.
- Proved that compactly supported vector fields are complete.
- Flows (1-PS) of diffeomorphisms.

Recap

- Defined integral curves and proved existence/uniqueness.
- Proved that compactly supported vector fields are complete.
- Flows (1-PS) of diffeomorphisms.
- Proved that the diffeo group acts transitively.

Another application of flows

Another application of flows

- What is an example of

Another application of flows

- What is an example of a local smooth vector field

Another application of flows

- What is an example of a local smooth vector field near a point?

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question:

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river,

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed?

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets,

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed.

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem:

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary).

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$.

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$. Then there exists a neighbourhood around p and

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$. Then there exists a neighbourhood around p and a coordinate chart s^i such that

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$. Then there exists a neighbourhood around p and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ in that neighbourhood.

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$. Then there exists a neighbourhood around p and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ in that neighbourhood.
- In a sense, s^1 functions as

Another application of flows

- What is an example of a local smooth vector field near a point? The simplest one is $\frac{\partial}{\partial x^1}$ for some coordinate x^1.
- A related physical question: Following a small paper boat in a river, can we somehow get a sense of the time elapsed? Of course, the farther the paper boat gets, the more the time has elapsed. However, what if the boat is placed at a point where the river isn't moving?
- Theorem: Let X be a smooth vector field on a smooth manifold M (without boundary). Suppose $X(p) \neq 0$. Then there exists a neighbourhood around p and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ in that neighbourhood.
- In a sense, s^1 functions as a "time coordinate".

Another application of flows

Proof:

Proof: In a coordinate neighbourhood (U, x) of p centred at p, Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin)

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \dots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \dots, x^n)$.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem,

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map.

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough,

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is

$(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?)

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo and hence $(s, x^2, \ldots, x^n)$ is a new coordinate chart around p.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo and hence $(s, x^2, \ldots, x^n)$ is a new coordinate chart around p. In this chart,

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo and hence $(s, x^2, \ldots, x^n)$ is a new coordinate chart around p. In this chart, the integral curve starting at $(0, x^2, \ldots, x^n)$ is

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo and hence $(s, x^2, \ldots, x^n)$ is a new coordinate chart around p. In this chart, the integral curve starting at $(0, x^2, \ldots, x^n)$ is $\gamma(t) = (t, x^2, \ldots, x^n)$.

Another application of flows

Proof: In a coordinate neighbourhood (U, x) of p centred at p, $X \neq 0$. WLog assume that $X^1 \neq 0$ on U. By the existence/uniqueness theorem, after shrinking U if necessary, there exists $\epsilon > 0$ such that there is an integral curve on $(-\epsilon, \epsilon)$ starting from any $q \in U$. Consider the map $F : (-\frac{\epsilon}{2}, \frac{\epsilon}{2}) \times V \to M$ (where V is some neighbourhood of the origin) given by $F(s, x^2, \ldots, x^n) = \gamma(s)$ where γ is the integral curve starting at $(0, x^2, \ldots, x^n)$. By the smooth dependence part of the existence theorem, F is a smooth map. Assuming V is small enough, we can assume that the image of F lies in U. The derivative of F is $(s = 0, 0, \ldots, 0)$ is $DF = \begin{bmatrix} X^1(p) & 0 & 0 & \ldots \\ X^2(p) & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ (why?), which is invertible (why?) Thus by IFT, F is a local diffeo and hence $(s, x^2, \ldots, x^n)$ is a new coordinate chart around p. In this chart, the integral curve starting at $(0, x^2, \ldots, x^n)$ is $\gamma(t) = (t, x^2, \ldots, x^n)$. Thus, $\gamma' = (1, 0, \ldots, 0)$ and $X = \frac{\partial}{\partial s}$.

Collar neighbourhood theorem (vague optional pearls of wisdom)

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted):

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary,

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0, 1) \times \partial M$ such that

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that using this theorem

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that using this theorem one can define the "double"

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that using this theorem one can define the "double" of a manifold-with-boundary.

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0, 1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that using this theorem one can define the "double" of a manifold-with-boundary.
- One can also define a

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that using this theorem one can define the "double" of a manifold-with-boundary.
- One can also define a connected sum of two manifolds

Collar neighbourhood theorem (vague optional pearls of wisdom)

- Collar neighbourhood theorem (proof omitted): If M is a smooth manifold-with-boundary, there is a neighbourhood of ∂M that is diffeomorphic to a "collar" $[0,1) \times \partial M$ such that ∂M goes to $\{0\} \times \partial M$.
- The point is that using this theorem one can define the "double" of a manifold-with-boundary.
- One can also define a connected sum of two manifolds by removing spheres and "gluing" them.

Lie bracket: motivation

Lie bracket: motivation

- Suppose we have

Lie bracket: motivation

- Suppose we have two vector fields X, Y

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p.

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart,

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes"

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields.

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square",

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?
- In other words,

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?
- In other words, $\gamma(s, p)$ and $\psi(t, p)$ are integral curves

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?
- In other words, $\gamma(s, p)$ and $\psi(t, p)$ are integral curves of X, Y resp. starting at p,

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?
- In other words, $\gamma(s, p)$ and $\psi(t, p)$ are integral curves of X, Y resp. starting at p, then how do we know that

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?
- In other words, $\gamma(s, p)$ and $\psi(t, p)$ are integral curves of X, Y resp. starting at p, then how do we know that $\gamma(-s, \psi(-t, \gamma(s, \psi(t, p)))) = p$?

Lie bracket: motivation

- Suppose we have two vector fields X, Y that are linearly independent at p. Can we say that there is a neighbourhood and a coordinate chart s^i such that $X = \frac{\partial}{\partial s^1}$ and $Y = \frac{\partial}{\partial s^2}$?
- If there was such a chart, the coordinate "axes" should have been obtained by flowing these vector fields. But how do we know that if we around a "coordinate square", we will come back to the same point?
- In other words, $\gamma(s, p)$ and $\psi(t, p)$ are integral curves of X, Y resp. starting at p, then how do we know that $\gamma(-s, \psi(-t, \gamma(s, \psi(t, p)))) = p$? At least, is $\frac{\partial^2 \gamma(-s,\psi(-t,\gamma(s,\psi(t,p))))}{\partial s \partial t}|_{s=t=0} = 0$?

Lie bracket: motivation

Lie bracket: motivation

- $\frac{\partial \gamma^i(-s,F(s,t))}{\partial s}|_{s=0} = -\frac{d\gamma^i}{ds}|_{s=0} + \frac{\partial \gamma^i}{\partial x^j}\frac{\partial F^j}{\partial s}|_{s=0} = -X^i(F(0,t)) + \frac{\partial F^i}{\partial s}|_{s=0}.$

Lie bracket: motivation

- $\frac{\partial \gamma^i(-s,F(s,t))}{\partial s}|_{s=0} = -\frac{d\gamma^i}{ds}|_{s=0} + \frac{\partial \gamma^i}{\partial x^j}\frac{\partial F^j}{\partial s}|_{s=0} = -X^i(F(0,t)) + \frac{\partial F^i}{\partial s}|_{s=0}$.
- Now $\frac{\partial F^j}{\partial s}|_{s=0} = \frac{\partial \psi^j}{\partial x^k}|_{s=0}\frac{\partial \gamma^k}{\partial s}|_{s=0} = \frac{\partial \psi^j}{\partial x^k}(-t,\psi(t,p))X^k(\psi(t,p))$.

Lie bracket: motivation

- $\frac{\partial\gamma^i(-s,F(s,t))}{\partial s}|_{s=0} = -\frac{d\gamma^i}{ds}|_{s=0} + \frac{\partial\gamma^i}{\partial x^j}\frac{\partial F^j}{\partial s}|_{s=0} = -X^i(F(0,t)) + \frac{\partial F^i}{\partial s}|_{s=0}$.
- Now $\frac{\partial F^j}{\partial s}|_{s=0} = \frac{\partial\psi^j}{\partial x^k}|_{s=0}\frac{\partial\gamma^k}{\partial s}|_{s=0} = \frac{\partial\psi^j}{\partial x^k}(-t,\psi(t,p))X^k(\psi(t,p))$.
- Taking one more derivative,

- $\frac{\partial\gamma^i(-s,F(s,t))}{\partial s}|_{s=0} = -\frac{d\gamma^i}{ds}|_{s=0} + \frac{\partial\gamma^i}{\partial x^j}\frac{\partial F^j}{\partial s}|_{s=0} = -X^i(F(0,t)) + \frac{\partial F^i}{\partial s}|_{s=0}.$
- Now $\frac{\partial F^j}{\partial s}|_{s=0} = \frac{\partial\psi^j}{\partial x^k}|_{s=0}\frac{\partial\gamma^k}{\partial s}|_{s=0} = \frac{\partial\psi^j}{\partial x^k}(-t,\psi(t,p))X^k(\psi(t,p)).$
- Taking one more derivative, we get $\frac{\partial^2\gamma(-s,\psi(-t,\gamma(s,\psi(t,p))))}{\partial s\partial t}|_{s=t=0} = -\frac{\partial X^i}{\partial x^j}(p)\frac{\partial F^j}{\partial t}|_{t=0} + \frac{\partial}{\partial t}\left(\frac{\partial\psi^i}{\partial x^k}(-t,\psi(t,p))X^k(\psi(t,p))\right)_{t=0},$

Lie bracket: motivation

- $\frac{\partial\gamma^i(-s,F(s,t))}{\partial s}|_{s=0} = -\frac{d\gamma^i}{ds}|_{s=0} + \frac{\partial\gamma^i}{\partial x^j}\frac{\partial F^j}{\partial s}|_{s=0} = -X^i(F(0,t)) + \frac{\partial F^i}{\partial s}|_{s=0}$.
- Now $\frac{\partial F^j}{\partial s}|_{s=0} = \frac{\partial\psi^j}{\partial x^k}|_{s=0}\frac{\partial\gamma^k}{\partial s}|_{s=0} = \frac{\partial\psi^j}{\partial x^k}(-t,\psi(t,p))X^k(\psi(t,p))$.
- Taking one more derivative, we get $\frac{\partial^2\gamma(-s,\psi(-t,\gamma(s,\psi(t,p))))}{\partial s\partial t}|_{s=t=0} = -\frac{\partial X^i}{\partial x^j}(p)\frac{\partial F^j}{\partial t}|_{t=0} + \frac{\partial}{\partial t}\left(\frac{\partial\psi^i}{\partial x^k}(-t,\psi(t,p))X^k(\psi(t,p))\right)_{t=0}$, which is $-\frac{\partial Y^i}{\partial x^k}(p)X^k(p) + \frac{\partial X^i}{\partial x^k}(p)Y^k(p)$.

Lie bracket: Definition

Lie bracket: Definition

- This last expression actually defines a vector field (why?)

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def:

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M.

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
- Lemma (proof by calculation):

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
- Lemma (proof by calculation): The Lie bracket genuinely defines a vector field

Lie bracket: Definition

- This last expression actually defines a vector field (why?)
- Is there an coordinate-invariant way of defining this vector field?
- Def: Let X, Y be smooth vector fields on a manifold M. Then $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$ is a vector field on M called the Lie bracket of X and Y.
- Lemma (proof by calculation): The Lie bracket genuinely defines a vector field whose components are given above.

Lie bracket: Properties

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$!

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with such a "bracket"

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with such a "bracket" is called a Lie algebra.

Lie bracket: Properties

- $[X, Y] = -[Y, X]$.
- $[X, Y]$ is multi-linear in X, Y.
- It is not associative! That is $[X, [Y, Z]] \neq [[X, Y], Z]$! Indeed, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi's identity).
- Any vector space equipped with such a "bracket" is called a Lie algebra. The space of smooth vector fields is an example.

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields,

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted):

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p,

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood and a coordinate chart

Lie bracket: Characterisation of coordinate vector fields

- If X, Y are coordinate vector fields, they Lie-commute (why?).
- Conversely, Theorem (proof omitted): If $X^1, X^2, \ldots, X^k$ are smooth Lie-commuting vector fields that are linearly independent at p, there is a neighbourhood and a coordinate chart such that $X^i = \frac{\partial}{\partial x^i}$.