MA 229/MA 235 - Lecture 27

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Recap

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- Saw that even integrating $x^2 dx \wedge dy$ over $\bar{D}$ is a tricky affair.

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