MA 229/MA 235 - Lecture 13

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- Proof: Choose some arbitrary charts centred at $p, F(p)$. Now the problem is a local one, i.e., if $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map with constant rank r and $F(0) = 0$, then we need to prove that there exist local diffeos $\phi : V \subset U \to \phi(V) \subset \mathbb{R}^m$ and $\psi : W \subset \mathbb{R}^n \to \psi(W)$ such that

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