MA 229/MA 235 - Lecture 5

IISc

Recap

Recap

- Second derivative test.

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- Inverse and implicit function theorems.

- Find the

An application - Lagrange's multipliers

- Find the maximum value of

An application - Lagrange's multipliers

- Find the maximum value of $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$.

An application - Lagrange's multipliers

- Find the maximum value of $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$.
- Geometrically,

An application - Lagrange's multipliers

- Find the maximum value of $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$.
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- Find the maximum value of $f(x, y, z) = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$.
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An application - Lagrange's multipliers

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Smooth structures and smooth manifolds

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