MA 229/MA 235 - Lecture 9

IISc

Recap

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- Manifolds (with or without boundary) have compact exhausations.

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- More examples of smooth maps.
- Manifolds (with or without boundary) have compact exhausations.
- Manifolds (with or without boundary) are paracompact.

Partition-of-unity

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- For every i, choose an index $a(i)$ so that $\bar{B}_i \subset U_{a(i)}$. For each α, define $\rho_\alpha = \sum_{a(i)=\alpha} g_i$. $supp(\rho_\alpha) = \cup_{i:a(i)=\alpha} \bar{B}_i \subset U_\alpha$. □

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  Proof:

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- Extension of smooth functions: Let $A \subset M$ be closed and $f : A \to \mathbb{R}^k$ be smooth.

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- The point of the tangent plane/tangent vectors is to provide a linear approximation to the manifold. Likewise, can we hope that tangent vectors can be deduced by knowing linear approximations of smooth functions?
- For instance, in $\mathbb{R}^n$, the linear approximation of a smooth function can be deduced if we know all directional derivatives. The directional derivative $D_{a,v}f = \frac{\partial f}{\partial x^i}(a)v^i$. So we can "read off" the components of tangent vectors from directional derivatives of smooth functions.

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- So what properties characterise directional derivatives?

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- $D_{a,v}$ is an example of a derivation.

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- Proof: Define the vector space structure as $(\alpha D_1 + \beta D_2)f = \alpha D_1 f + \beta D_2 f$.

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- Corollary: The derivations $\frac{\partial}{\partial x^i}|_a$ defined by $\frac{\partial}{\partial x^i}|_a f = \frac{\partial f}{\partial x^i}(a)$ form a basis for $T_a\mathbb{R}^n$.