MA 229/MA 235 - Lecture 15

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Recap

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- Implicit, inverse, constant rank theorems. Regular values and Sard's theorem.
- Partitions-of-unity, Whitney's embedding.

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- A natural choice is the coordinate basis $\frac{\partial}{\partial x^i}$.

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